Monotone tail functions: Definitions, properties, and application to risk-reducing strategies

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ABSTRACT

This paper studies properties of functions having monotone tails. We extend Theorem 1 of Dhaene et al. (2002a) and show how the tail quantiles of a random variable transformed with a monotone tail function can be expressed as the transformed tail quantiles of the original random variable. The main result is intuitive, in that Dhaene et al. (2002a)'s properties still hold, but only for certain quantile values. However, the proof presents some complications that arise especially when the function involved has discontinuities.

We consider different situations where monotone tail functions occur and can be useful, such as the evaluation of the payoff of option trading strategies and the present value of insurance contracts providing both death and survival benefits. The paper also applies monotone tail functions to study quadrant perfect dependence, and shows how this dependence structure integrates within the framework of monotone tail functions. Moreover, we apply the theory to the problem of risk reduction and investigate conditions on a hedger ensuring efficient reductions of required economic capital.

1. Introduction

A standard result derived in [1] expresses the quantile of a random variable transformed by a monotone function as the transform of the original random variable's quantile. The present paper derives sufficient conditions such that this result remains to hold when the transformation involves a function which is not monotone. We extend Theorem 1 of Dhaene et al. [1] by investigating *monotone tail* functions h: functions which are monotone after or before a certain threshold. We show in this paper that if h has a monotone tail, then transforming the original random variable using h and then taking the quantile of the transformed random variable, is equal to first taking the quantile and then transforming this quantile using the function h, provided the confidence level p of the quantile is within an explicitly given interval. This result complies with the intuition, but the proof presents some challenges when the function h has discontinuities.

The paper studies different situations in which such functions having a monotone tail arise. Option trading strategies involving combinations of calls and puts are typical examples where the payoff function of the strategy has a monotone tail. These strategies are often used to neutralize portfolios in some parts of the payoff's support. Determining the quantiles in the tail of the payoff's distribution can be achieved using the theory developed in this paper. We also show that monotone tail functions arise in the distribution of the payout of insurance contracts such as endowment assurances.

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These contracts pay a benefit in case of death or survival of the policyholder. Thus, the present value of the benefit is expressed as the sum of a non-decreasing function and a non-increasing function of the remaining lifetime, and the resulting function has a monotone tail.

A related concept is *quadrant perfect dependence*. For example, upper comonotonicity, which was introduced in [2], refers to a perfect positive dependence structure in the upper quadrant of their joint support; see also [3–5]. In general, quadrant perfect dependence refers to a dependence structure under which two random variables are perfectly positive or negative dependent in a quadrant. The possible generalization of upper comonotonicity to other quadrants has already been discussed in e.g. [3]. The present paper studies these dependence structures through the lens of the proposed monotone tail functions. We show that monotonicity in the tail of the difference of the two quantile functions of two comonotonic random variables R_1 and R_2 is equivalent with quadrant perfect dependence of the random vector $(R_2, R_1 - R_2)$.

The paper also employs monotone tail functions and quadrant perfect dependence to address the problem of selecting an appropriate Value-at-Risk reducer. Risk reduction consists in finding an appropriate asset R_2 whose interaction with a given liability R_1 is such that the risk of the residual liability $R_1 - R_2$ is reduced; see [6,7] for some early studies. In terms of solvency capital, an insurance company faces the problem of selecting a hedger R_2 such that the amount of required regulatory capital is reduced, where this capital is measured using an appropriate quantile, i.e. Value-at-Risk (VaR). Intuitively, a candidate risk reducer R_2 should be (strongly) positively dependent with R_1. But even when R_1 and R_2 are perfectly positively dependent (i.e. comonotonic), selecting an adequate asset to reduce the risk of the liabilities is not a straightforward task. We work under a setting where R_1 and R_2 are comonotonic random variables, i.e. both random variables are non-decreasing functions of the same random source. This assumption provides a useful proxy when studying risk reducers; see also [6,8]. In this case, the difference $Z = R_1 - R_2$ can be expressed as $Z \stackrel{d}{=} h(U)$, where U is a random variable uniformly distributed over the unit interval and

$$h(p) = F_{R_1}^{-1}(p) - F_{R_2}^{-1}(p). \tag{1.1}$$

This equality does not necessarily mean that the VaR of the residual liability Z is the difference between the VaR of the initial position and that of the candidate VaR-reducer. Theorem 1 of Dhaene et al. [1] ensures that such a decomposition holds when the function h is monotone. In our case, however, the function h is not necessarily monotone. As a consequence, we cannot directly express the quantile of the difference as a difference of quantiles. When using the VaR to determine solvency capital, one is mainly interested in quantiles with sufficiently high level p. The theory on functions having a monotone tail is illustrated by showing that the decomposition of the quantile of Z holds if we limit the range of p to some interval of interest, especially in the tail. Alternatively, it should be acknowledged that the decomposition of the quantile of the difference can be derived using the theory on upper comonotonicity and the extension of quadrant perfect dependence. Either way, this decomposition allows to derive conditions on the choice of an appropriate VaR-reducer. This leads to the conclusion that if the upper tail of the residual liability is dominated by the initial liability, the intended reduction in solvency capital is achieved provided the asset R_2 is traded at a reasonable price. In the opposite case, a reasonable price for R_2 might not exist.

In order to gain more insights into the comonotonic risk mitigation process, we consider a general setting where R_1 and R_2 have equally distributed standardized log-returns. We show in this case that the support of the residual liability always has a monotone upper tail. We also find that the price conditions for VaR reduction depend on the relative standard deviations of the candidate VaR reducer and the initial liability. On the one hand, in case the standard deviation associated with R_2 is smaller than that of the initial liability, we find that R_2 is a VaR-reducer provided its price is smaller than its VaR at the confidence level p. Since in practice p takes high values, this condition will be satisfied for any reasonable price of R_2. On the other hand, in case the standard deviation associated with the candidate risk reducer is larger than that of the initial liability, we find that the losses from the residual liability are bounded, and it can still occur that R_2 is not a VaR-reducer. What explains this result is that in such situations, the price of the candidate hedger is too large compared to the capital relief, and the capital required to achieve the risk reduction exceeds the capital needed to cover the liability R_1 when it is not hedged at all.

Comonotonicity of the initial liability and the risk reducer implies that the risk reduction is efficient, in the sense that a large loss is followed by a large payoff of the risk reducer. However, if the payoff from the risk reducer is too large, the initial claim is over-hedged, which can potentially be costly and therefore not necessarily desirable. One situation which avoids this over-hedging is when the support of the residual loss has a non-decreasing upper tail. This ensures that if the risk reducer is large, the residual loss will also increase. In other words, the risk reducer does not dominate the initial liability in the upper tail. As we show in this paper, this is equivalent with the situation where the risk reducer and the residual liabilities are upper comonotonic.

An alternative, and more general, characterization of risk mitigation was given in [8], who define the notion of a risk reducer in terms of the Tail Value-at-Risk (TVaR). They show that if R_1 and R_2 move in the same direction, and hence R_2 would presumably compensate the high losses of R_1, the hedged portfolio is not necessarily less risky than the initial position. In other words, there remain situations where not hedging at all is better than hedging with a comonotonic asset. Nevertheless, when the asset and the residual liability are comonotonic, then this asset is a risk reducer in the sense of Cheung et al. [8]. We also refer to He et al. [9] for a generalization to risk reducers which are not perfectly dependent.

Our analysis shows that the conclusions on VaR-reducers hold for TVaR-reducers too, if we limit the condition of TVaR reduction to some values of the confidence level.

The remainder of the paper is organized as follows. Section 2 contains preliminary definitions and notations. Section 3 provides the main technical results of the paper, with the definitions of all four types of monotone tail functions and the theorem extending that of Dhaene et al. [1] in each case. We study monotone tail functions in the context of option trading strategies and endowment assurances in Section 4. Four types of quadrant perfect dependence are discussed and their relationship with monotone tail functions is established in Section 5. We address in Section 6 the problem of risk reduction using monotone tail functions, and we investigate conditions on the distribution of R_2 leading to a VaR reduction. We conclude in Section 7. All proofs can be found in the Appendix.

2. Preliminaries

All random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and can be discrete or continuous. The cumulative distribution function (cdf) of a random variable X is denoted by F_X, where X takes values in the real interval $\mathcal{I} \subseteq \mathbb{R}$. The left inverse cdf F_X^{-1} of X is defined as

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0, 1], \tag{2.1}$$

with $\inf \emptyset = +\infty$, by convention, and the Value-at-Risk of X at the level p, denoted by $\mathrm{VaR}_p[X]$, corresponds to $F_X^{-1}(p)$. The Tail Value-at-Risk at the level p is defined as follow:

$$\mathrm{TVaR}_p[X] = \frac{1}{1-p} \int_p^1 F_X^{-1}(q)\mathrm{d}q.$$

The Left TVaR at the level p is defined as follows:

$$\mathrm{LTVaR}_p[X] = \frac{1}{p} \int_0^p F_X^{-1+}(q)\mathrm{d}q,$$

where the right inverse F_X^{-1+} is defined as follows:

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}, \qquad p \in [0, 1], \tag{2.2}$$

where $\sup \emptyset = -\infty$, by convention. We also use the generalized α-inverse $F_X^{-1(\alpha)}$:

$$F_X^{-1(\alpha)}(p) = (1-\alpha)F_X^{-1}(p) + \alpha F_X^{-1+}(p), \qquad p \in [0, 1] \text{ and } \alpha \in [0, 1], \tag{2.3}$$

More details on generalized inverses can be found in [10].

A set $\mathcal{S}$ is comonotonic if for any couples $(x_1, x_2), (y_1, y_2) \in \mathcal{S}$, we have that $x_1 \leq y_1$ and $x_2 \leq y_2$, or $x_1 \geq y_1$ and $x_2 \geq y_2$. More specifically, a comonotonic set is non-decreasing in both its components. A random vector (X_1, X_2) is said to be comonotonic if its support is a comonotonic set. Analogously, a random vector is said to be counter-monotonic if its support is a counter-monotonic set, where a set $\mathcal{S}$ is counter-monotonic if for any $(x_1, x_2), (y_1, y_2) \in \mathcal{S}$, we have that $x_1 \leq y_1$ and $x_2 \geq y_2$, or $x_1 \geq y_1$ and $x_2 \leq y_2$.

3. Quantiles of random variables transformed with monotone tail functions

3.1. Monotone functions

Consider a function $h : \mathcal{I} \mapsto \mathcal{J}$, with $\mathcal{I}, \mathcal{J} \subseteq \mathbb{R}$. Dhaene et al. [1] derive conditions such that the quantiles of a transformed random variable coincide with the transformed quantiles of the original random variable. This is formally stated in the following theorem.

Theorem 3.1. *Consider the random variable X with cdf F_X, and a function $h : \mathcal{I} \mapsto \mathcal{J}$.*

1. *If h is non-decreasing and left-continuous, then:*

$$F_{h(X)}^{-1}(p) = h\left(F_X^{-1}(p)\right), \qquad \text{for } p \in (0, 1). \tag{3.1}$$

2. *If h is non-decreasing and right-continuous, then:*

$$F_{h(X)}^{-1+}(p) = h\left(F_X^{-1+}(p)\right), \qquad \text{for } p \in (0, 1). \tag{3.2}$$

3. *If h is non-increasing and left-continuous, then:*

$$F_{h(X)}^{-1+}(p) = h\left(F_X^{-1}(1-p)\right), \qquad \text{for } p \in (0, 1). \tag{3.3}$$

(a) Quadrants based on the point $(u, h(u))$.

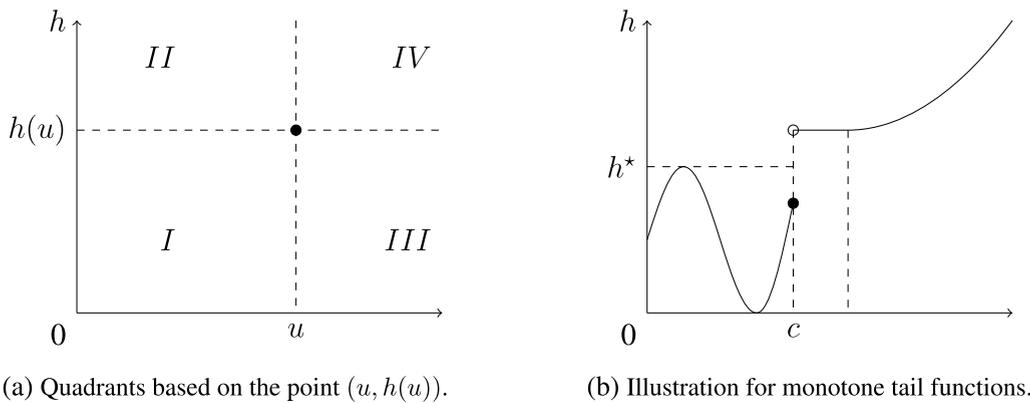(b) Illustration for monotone tail functions.

Fig. 1. Functions with a non-decreasing upper tail.

4. If h is non-increasing and right-continuous, then:

$$F_{h(X)}^{-1}(p) = h\left(F_X^{-1+}(1-p)\right), \qquad for\ p \in (0, 1). \tag{3.4}$$

In this section, it is shown that, for some values of p, the results of Theorem 3.1 hold even in case h is not monotone everywhere. The functions under interest are monotone after or before a certain threshold. Such functions are said to have a monotone tail. It is proven that Section 3 remains to hold for these functions, provided the probabilities p are either sufficiently large, or sufficiently small.

3.2. Monotone tail functions

Generally speaking, a function $h : \mathcal{I} \mapsto \mathcal{J}$ is said to have a monotone tail if $\mathcal{I} \times \mathcal{J}$ can be divided into four quadrants (see Fig. 1(a)), such that h is monotone in one of the quadrants. Naturally, there are four types of monotone tail functions. The function h can be either non-decreasing or non-increasing *after* a certain threshold. Such functions are said to have a monotone upper, or right, tail. Analogously, the function h can be either non-decreasing or non-increasing *before* a certain threshold. Such functions are said to have a monotone lower, or left, tail.

In the sequel, the function used to transform the random variable X is assumed to be real-valued, and can only have right or left continuous jumps. The notation h is used for functions with non-decreasing tail, and g for functions with non-increasing tail. Further, the subscripts "u" and "l" are used to indicate whether the monotonicity is in the upper or the lower tail, respectively. The initial definition is that of functions having a non-decreasing upper tail.

Definition 3.1 (*Non-decreasing Upper Tail*)**.** The function h_u is said to have a *non-decreasing upper tail* if there exists a constant $x' \in \mathcal{I}$ such that:

1. $h_u(x) \leq h_u(x')$, for all elements of the set $\{x \in \mathcal{I} | x < x'\}$, provided it is not empty,
2. $h_u(x_1) \leq h_u(x_2)$, for all $x_1, x_2 \in \mathcal{I}$ with $x' \leq x_1 \leq x_2$.

The function h_u is said to have a *non-increasing upper tail with threshold* $c_u = \inf \mathcal{H}_u$, where $\mathcal{H}_u$ contains all x' satisfying Conditions (1) and (2).

Condition (1) of Definition 3.1 states that the function h_u only takes values in quadrants I and IV of Fig. 1(a), and does not enter the quadrants II and III. Condition (2) of Definition 3.1 states that the function should be non-decreasing in quadrant IV.

Note that when h_u is non-decreasing, then $\mathcal{H}_u = \mathcal{I}$. In this case, it is sufficient to use the result of Dhaene et al. [1] stated in Theorem 3.1. When the function h_u does not have a non-decreasing upper tail, the set $\mathcal{H}_u$ is empty, and by convention, $c_u = \sup \mathcal{I}$. In this case, monotone tail functions are not relevant. Thus, throughout this paper, these trivial cases are ruled out, and two assumptions are used, namely, $\mathcal{H}_u$ is not empty, and $c_u \in (\inf \mathcal{I}, \sup \mathcal{I})$.

The other forms of monotone tail functions are now defined.

Definition 3.2 (*Non-increasing Upper Tail*)**.** The function g_u is said to have a *non-increasing upper tail* if there exists $x' \in \mathcal{I}$ such that:

1. $g_u(x) \geq g_u(x')$, for all elements of the set $\{x \in \mathcal{I} | x < x'\}$, provided it is not empty,
2. $g_u(x_1) \geq g_u(x_2)$, for all $x_1, x_2 \in \mathcal{I}$ with $x' \leq x_1 \leq x_2$.

The function g_u is said to have a *non-increasing upper tail with threshold* $d_u = \inf \mathcal{G}_u$, where $\mathcal{G}_u$ contains all x' satisfying Conditions (1) and (2).

Definition 3.3 (*Non-decreasing Lower Tail*). The function h_l is said to have a *non-decreasing lower tail* if there exists $x' \in \mathcal{I}$ such that:

1. $h_l(x) \geq h_l(x')$, for all elements of the set $\{x \in \mathcal{I} \mid x > x'\}$, provided it is not empty,
2. $h_l(x_1) \leq h_l(x_2)$, for all $x_1, x_2 \in \mathcal{I}$ with $x_1 \leq x_2 \leq x'$.

The function h_l is said to have a *non-decreasing lower tail with threshold* $c_l = \sup \mathcal{H}_l$, where $\mathcal{H}_l$ contains all x' satisfying Conditions (1) and (2).

Definition 3.4 (*Non-increasing Lower Tail*). The function g_l is said to have a *non-increasing lower tail* if there exists $x' \in \mathcal{I}$ such that:

1. $g_l(x) \leq g_l(x')$, for all elements of the set $\{x \in \mathcal{I} \mid x > x'\}$, provided it is not empty,
2. $g_l(x_1) \geq g_l(x_2)$, for all $x_1, x_2 \in \mathcal{I}$ with $x_1 \leq x_2 \leq x'$.

The function g_l is said to have a *non-increasing lower tail with threshold* $d_l = \sup \mathcal{G}_l$, where $\mathcal{G}_l$ contains all x' satisfying Conditions (1) and (2).

Before we proceed, let us make two remarks regarding monotone tail functions.

Remark 3.2. A non-decreasing (resp. non-increasing) function has both a non-decreasing (resp. non-increasing) lower tail, and a non-decreasing (resp. non-increasing) upper tail.

Remark 3.3. If a function has a monotone lower tail (resp. upper tail), then it can only have a monotone upper tail (resp. lower tail).

In order to illustrate Remark 3.3, consider a function h having a non-decreasing lower tail. This means that h is a non-decreasing injection on Quadrant I of Fig. 1(a), and cannot take values in Quadrant II or Quadrant III. Thus, h can only take values in Quadrant IV, implying that it can only have a monotone upper tail.

The following lemma provides a link between the four different forms of monotone tail functions, and allows to focus on one of them in the proof of the main result. Throughout, for any set $\mathcal{D}$, the notation $\bar{\mathcal{D}}$ refers to the transformation of the elements of $\mathcal{D}$ such that:

$$x \in \mathcal{D} \iff -x \in \bar{\mathcal{D}}. \tag{3.5}$$

Lemma 3.1. *Consider the function* $h_u : \mathcal{I} \mapsto \mathcal{J}$ *which has a non-decreasing upper tail with threshold* c_u, *and the sets* $\bar{\mathcal{J}}$ *and* $\bar{\mathcal{I}}$ *defined according to* (3.5). *The following holds:*

- *The function* $g_u : \mathcal{I} \mapsto \bar{\mathcal{J}}$, *such that* $g_u(x) = -h_u(x)$ *has a non-increasing upper tail with threshold* c_u.
- *The function* $h_l : \bar{\mathcal{I}} \mapsto \bar{\mathcal{J}}$, *such that* $h_l(x) = -h_u(-x)$ *has a non-decreasing lower tail with threshold* $-c_u$.
- *The function* $g_l : \bar{\mathcal{I}} \mapsto \mathcal{J}$, *such that* $g_l(x) = h_u(-x)$ *has a non-increasing lower tail with threshold* $-c_u$.

3.3. Intermediary results

In what follows, functions having non-decreasing upper tail are the main focus. Later, the links derived in Lemma 3.1 are used to extend the results to all four types of monotone tail functions. The subscript "u" is momentarily dropped, such that a function having a non-decreasing upper tail with threshold c is denoted by h.

In order to prove our main result, the function h is transformed into a non-decreasing function $\tilde{h}$. This transformed function is such that it coincides with the original function h in the upper part exceeding c, whereas the lower part is constructed such that the function $\tilde{h}$ is non-decreasing. However, moving back from the setting of $\tilde{h}$ to that of h raises some challenges, especially in case of discontinuity at c. Circumventing those challenges requires some preliminary steps, to which the present subsection is devoted.

Define the quantity $h^\star$ as follows:

$$h^\star = \sup_{x < c} h(x), \tag{3.6}$$

The function h only takes values exceeding $h^\star$ if $x \geq c$. The value associated with c used to determine the quadrants is given by $h^\star$.

Define also the function $\tilde{h} : \mathcal{I} \mapsto \mathcal{J}$ as follows:

$$\tilde{h}(x) = \begin{cases} h^\star, & \text{if } x < c, \\ \max\{h(c), h^\star\}, & \text{if } x = c, \\ h(x), & \text{if } x > c. \end{cases} \tag{3.7}$$

The new function $\tilde{h}$ is non-decreasing on the whole interval $\mathcal{I}$.

Finally, define the probability π_c as follows

$$\pi_c = \mathbb{P}\left[h(X) \leq h^\star\right].$$

Since c is the infimum of the set $\mathcal{H}$, it follows from Condition (1) of Definition 3.1 that $X < c$ implies $h(X) \leq h^\star$. However, $h(X) \leq h^\star$ does not necessarily imply that $X \leq c$, especially if $h^\star$ corresponds to a horizontal part. Thus, the following inequality holds:

$$\pi_c \geq \mathbb{P}\left[X < c\right],$$

and becomes an equality if h is strictly increasing and continuous in c.

Four important lemmas are now stated and will be used for the proof of the main theorem; see Appendices A.2, A.3, A.4 and A.5 for the proofs. The first lemma states that transforming a random variable using h or $\tilde{h}$ does not affect the resulting cdf after the threshold $h^\star$.

Lemma 3.2. *Consider a function h having a non-decreasing upper tail with threshold c, and the function $\tilde{h}$ defined in (3.7). For $\pi_c = \mathbb{P}\left[h(X) \leq h^\star\right]$ and $h^\star = \sup_{x<c} h(x)$, it follows that:*

$$\mathbb{P}\left[\tilde{h}(X) \leq y\right] = \mathbb{P}\left[h(X) \leq y\right], \qquad \text{for } y \geq h^\star, \tag{3.8}$$

and in particular:

$$\mathbb{P}\left[\tilde{h}(X) \leq h^\star\right] = \pi_c. \tag{3.9}$$

The second lemma states that the generalized quantile of the random variables $h(X)$ and $\tilde{h}(X)$ coincide when the probability level p is taken sufficiently large.

Lemma 3.3. *Consider a function h having a non-decreasing upper tail with threshold c, and the function $\tilde{h}$ defined in (3.7). For $\pi_c = \mathbb{P}\left[h(X) \leq h^\star\right]$ and $h^\star = \sup_{x<c} h(x)$, it follows that:*

$$F_{\tilde{h}(X)}^{-1(\alpha)}(p) = F_{h(X)}^{-1(\alpha)}(p), \qquad \text{for } p > \pi_c \text{ and } \alpha \in [0, 1]. \tag{3.10}$$

The third lemma states that the same value is obtained regardless of whether the generalized quantile $F_X^{-1(\alpha)}(p)$ is transformed using $\tilde{h}$ or h, provided the value p is sufficiently large.

Lemma 3.4. *Consider a function h having a non-decreasing upper tail with threshold c, and the function $\tilde{h}$ defined in (3.7). For $\pi_c = \mathbb{P}\left[h(X) \leq h^\star\right]$, it follows that:*

$$\tilde{h}\left(F_X^{-1(\alpha)}(p)\right) = h\left(F_X^{-1(\alpha)}(p)\right), \qquad \text{for } p > \pi_c \text{ and } \alpha \in [0, 1]. \tag{3.11}$$

The fourth lemma shows that left/right continuity of the function h is inherited by $\tilde{h}$.

Lemma 3.5. *Consider a function h having a non-decreasing upper tail with threshold c, and the function $\tilde{h}$ defined in (3.7). It follows that:*

1. *h is left-continuous $\Rightarrow$ $\tilde{h}$ is left-continuous.*
2. *h is right-continuous $\Rightarrow$ $\tilde{h}$ is right-continuous.*

3.4. Main theorem

Using these lemmas, together with Lemma 3.1 that links the four types of monotone tail functions, it is now possible to derive conditions under which the inverse of a transformed random variable $h(X)$ can be expressed as the function h applied on the inverse of the initial random variable X. The results are summarized in Theorem 3.4 and the proof can be found in Appendix A.6.

Theorem 3.4. *Consider a random variable X with cdf F_X.*

- *For a function h_u having a non-decreasing upper tail with threshold c_u, with $\pi_{c_u} = \mathbb{P}\left[h_u(X) \leq h_u^\star\right]$, and $h_u^\star = \sup_{x<c_u} h_u(x)$, the following holds:*

1. If h_u is left-continuous, then:

$$F_{h_u(X)}^{-1}(p) = h_u\left(F_X^{-1}(p)\right), \qquad \text{for } p > \pi_{c_u}. \tag{3.12}$$

2. If h_u is right-continuous, then:

$$F_{h_u(X)}^{-1+}(p) = h_u\left(F_X^{-1+}(p)\right), \qquad \text{for } p \geq \pi_{c_u}. \tag{3.13}$$

- For a function g_u having a non-increasing upper tail with threshold d_u, with $\lambda_{d_u} = \mathbb{P}\left[g_u(X) \geq g_u^\star\right]$, and $g_u^\star = \inf_{x < d_u} g_u(x)$, the following holds:

  1. If g_u is left-continuous, then:

  $$F_{g_u(X)}^{-1+}(p) = g_u\left(F_X^{-1}(1-p)\right), \qquad \text{for } p < 1 - \lambda_{d_u}. \tag{3.14}$$

  2. If g_u is right-continuous, then:

  $$F_{g_u(X)}^{-1}(p) = g_u\left(F_X^{-1+}(1-p)\right), \qquad \text{for } p \leq 1 - \lambda_{d_u}. \tag{3.15}$$

- For a function h_l having a non-decreasing lower tail with threshold c_l, with $\lambda_{c_l} = \mathbb{P}\left[h_l(X) \geq h_l^\star\right]$, and $h_l^\star = \inf_{x > c_l} h_l(x)$, the following holds:

  1. If h_l is left-continuous, then:

  $$F_{h_l(X)}^{-1}(p) = h_l\left(F_X^{-1}(p)\right), \qquad \text{for } p \leq 1 - \lambda_{c_l}. \tag{3.16}$$

  2. If h_l is right-continuous, then:

  $$F_{h_l(X)}^{-1+}(p) = h_l\left(F_X^{-1+}(p)\right), \qquad \text{for } p < 1 - \lambda_{c_l}. \tag{3.17}$$

- For a function g_l having a non-increasing lower tail with threshold d_l, with $\pi_{d_l} = \mathbb{P}\left[g_l(X) \leq g_l^\star\right]$, and $g_l^\star = \sup_{x > d_l} g_l(x)$, the following holds:

  1. If g_l is left-continuous, then:

  $$F_{g_l(X)}^{-1+}(p) = g_l\left(F_X^{-1}(1-p)\right), \qquad \text{for } p \geq \pi_{d_l}. \tag{3.18}$$

  2. If g_l is right-continuous, then:

  $$F_{g_l(X)}^{-1}(p) = g_l\left(F_X^{-1+}(1-p)\right), \qquad \text{for } p > \pi_{d_l}. \tag{3.19}$$

Note that Expression (3.12) holds for values p which are greater or equal to π_c. Expression (3.13), on the other hand, only holds for values p which are strictly larger than π_c. It is possible to construct counter-examples to prove that in general:

$$F_{h_u(X)}^{-1}\left(\pi_{c_u}\right) \neq h_u\left(F_X^{-1}\left(\pi_{c_u}\right)\right),$$

for a function h_u having a non-decreasing upper tail with threshold c_u. This is the case for a random variable X with probability mass function:

$$\mathbb{P}[X = k] = \frac{1}{4}, \text{ for } k = 0, 1, 2, 3$$

and a function $h_u : [0, 3] \to \mathbb{R}$ given by:

$$h_u(x) = \begin{cases} x, & \text{if } x \in [0, 1], \\ 2 - x, & \text{if } x \in (1, 2], \\ 2(x - 2) & \text{if } x \in (2, 3]. \end{cases}$$

This remark is valid for the analogous expressions for the other types of monotone tail functions.

4. Examples and associated results

4.1. Option trading strategies

Option trading strategies provide numerous examples where the random variable denoting the payoff of the strategy can be expressed using the stock price transformed by a monotone tail function. Such strategies are combinations of calls and puts, and sometimes the stock price itself, and are used in certain situations to construct neutral portfolios [11].

A common example is the *straddle* option strategy. It consists in purchasing simultaneously a call option and a put option on the same underlying stock price, with the same strike and maturity. A straddle strategy is profitable especially when the underlying stock price rises. It is then relevant to determine the quantiles of the payoff in the upper tail.

For a given strike K, the payoff function of the straddle strategy is given by:

$$h(x) = \max\{K - x; 0\} + \max\{x - K; 0\},$$

with $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$. The function h has a non-decreasing upper tail with threshold $2K$. Moreover, $h^* = K$. Therefore, for a stock price at maturity denoted by X, the payoff at maturity is given by $h(X)$, and the corresponding quantile at the level p has the following expression:

$$F_{h(X)}^{-1}(p) = \max\left\{K - F_X^{-1}(p); 0\right\} + \max\left\{F_X^{-1}(p) - K; 0\right\}, \qquad \text{for } p > \mathbb{P}\left[h(X) \le K\right].$$

Many other option strategies are characterized by a payoff function having a monotone tail. The list includes the *strangle* strategy, whose difference with the straddle is that the strike of the call option is higher than that of the put option, The payoff function in this case again has a non-decreasing upper tail. The payoff of the *back spread with calls* (selling a call with strike K_1 and buying two calls with strike $K_2 > K_1$) also has a non-decreasing upper tail. In contrast, the short versions of the straddle and the strangle involve non-increasing upper tails. The *back spread with puts* (buying two puts with strike K_1 and selling a put with strike $K_2 > K_1$) is an example of strategies whose payoff function has a non-increasing lower tail.

4.2. Endowment assurances

Consider an endowment assurance contract, with a benefit amount $S_1 > 0$ payable upon death if it occurs before the term n, or a maturity benefit $S_2 > 0$ if the policyholder survives the term. Suppose that the death and survival benefits are not equal, i.e. $S_1 \ne S_2$, which is not an uncommon feature of endowment assurances, and is the case under double endowment assurances for instance. Let X be the random complete remaining lifetime of the policyholder. Assuming a constant discounting factor $v \in (0, 1)$ throughout the term, the random present value of this benefit is $h(X)$, with $h : \mathbb{R}^+ \mapsto \mathbb{R}^+$, where:

$$h(x) = S_1 v^x \mathbb{I}[x < n] + S_2 v^n \mathbb{I}[x \ge n],$$

and $\mathbb{I}$ is the indicator function. Note that the function h is constant for $x \ge n$. Moreover, h is right continuous at n, and has a jump at n for $S_1 \ne S_2$.

If $S_1 \le S_2 v^n$, the function h has a non-decreasing upper tail with threshold n, and hence:

$$F_{h(X)}^{-1+}(p) = S_1 v^{F_X^{-1+}(p)} \mathbb{I}\left[F_X^{-1+}(p) < n\right] + S_2 v^n \mathbb{I}\left[F_X^{-1+}(p) \ge n\right], \qquad \text{for } p \ge \mathbb{P}\left[h(X) \le S_1\right].$$

If $S_2 v^n < S_1$, the function g has a non-increasing lower tail with threshold $\frac{\log(S_2 v^n) - \log(S_1)}{\log(v)}$, and hence:

$$F_{h(X)}^{-1}(p) = S_1 v^{F_X^{-1+}(1-p)} \mathbb{I}\left[F_X^{-1+}(1 - p) < n\right] + S_2 v^n \mathbb{I}\left[F_X^{-1+}(1 - p) \ge n\right], \qquad \text{for } p > \mathbb{P}\left[h(X) \le S_2 v^n\right].$$

5. Quadrant perfect dependence

Cheung [2] introduced the concept of upper comonotonicity to model random variables which are perfectly positively dependent beyond a threshold of the joint support. More specifically, a random vector which is comonotonic in the top-right quadrant of the joint support is said to be upper comonotonic. Similar to comonotonic vectors, an upper–upper comonotonic vector has desirable additive properties for the Value-at-Risk of the sum of its components (see Proposition 6 in [2]). In this section, we investigate how this concept integrates within the framework of monotone tail functions.

The four different types of monotone tail functions call for the introduction of three other dependence structures. These dependence structures were mentioned in e.g. [3], and we label them *lower–lower comonotonicity* (i.e. perfect positive dependence in the bottom-left quadrant), *upper–lower counter-monotonicity* (i.e. perfect negative dependence in the bottom-right quadrant), and *lower–upper counter-monotonicity* (i.e. perfect negative dependence in the top-left quadrant). In order to avoid confusion, we use the term *upper–upper comonotonicity* when we refer to Cheung [2]'s dependence structure. The choice of this terminology is motivated by the following reasoning. For instance, an upper–lower counter-monotonic vector (X_1, X_2) is counter-monotonic in the quadrant where X_1 is in the upper part of its univariate support and X_2 is in the lower part of its univariate support.

Throughout this section, we use the following notations for the four quadrants at $\underline{a} = (a_1, a_2) \in \mathbb{R}^2$, where the boundary is excluded:

$$BL\left(\underline{a}\right) = (-\infty, a_1) \times (-\infty, a_2),$$
$$BR\left(\underline{a}\right) = (a_1, +\infty) \times (-\infty, a_2),$$
$$TL\left(\underline{a}\right) = (-\infty, a_1) \times (a_1, +\infty),$$
$$TR\left(\underline{a}\right) = (a_1, +\infty) \times (a_2, +\infty).$$

In particular, $BL(\underline{a})$ is the bottom-left quadrant, $BR(\underline{a})$ is the bottom-right quadrant, $TL(\underline{a})$ is the top-left quadrant, and $TR(\underline{a})$ is the top-right quadrant. We also use the bar to indicate that the boundary is included:

$$\overline{BL}\left(\underline{a}\right) = (-\infty, a_1] \times (-\infty, a_2],$$

$$\overline{BR}\left(\underline{a}\right) = [a_1, +\infty) \times (-\infty, a_2],$$
$$\overline{TL}\left(\underline{a}\right) = (-\infty, a_1] \times [a_2, +\infty),$$
$$\overline{TR}\left(\underline{a}\right) = [a_1, +\infty) \times [a_2, +\infty).$$

Definition 5.1 (*Upper–upper Comonotonicity*). A two-dimensional random vector $\underline{X}$ with support $\mathcal{S}$ is upper–upper comonotonic with threshold $\underline{a}$ if the following conditions hold:

1. $\mathcal{S}_{\underline{a}} = \mathcal{S} \cap TR(\underline{a})$ is a comonotonic set,
2. $\mathbb{P}\left[\underline{X} > \underline{a}\right] > 0$,
3. $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right)$ is empty.

Definition 5.2 (*Lower–lower Comonotonicitiy*). A two-dimensional random vector $\underline{X}$ with support $\mathcal{S}$ is lower–lower comonotonic with threshold $\underline{a}$ if the following conditions hold:

1. $\mathcal{S}_{\underline{a}} = \mathcal{S} \cap BL(\underline{a})$ is a comonotonic set,
2. $\mathbb{P}\left[\underline{X} < \underline{a}\right] > 0$,
3. $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(BR(\underline{a}) \cup \overline{TR}(\underline{a})\right)\right)$ is empty.

Definition 5.3 (*Upper–Lower Counter-monotonicity*). A two-dimensional random vector $\underline{X}$ with support $\mathcal{S}$ is upper–lower counter-monotonic with threshold $\underline{a}$ if the following conditions hold:

1. $\mathcal{S}_{\underline{a}} = \mathcal{S} \cap BR(\underline{a})$ is a counter-monotonic set,
2. $\mathbb{P}[X_1 > a_1, X_2 < a_2] > 0$,
3. $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(BR(\underline{a}) \cup \overline{TL}(\underline{a})\right)\right)$ is empty.

Definition 5.4 (*Lower–Upper Counter-monotonicity*). A two-dimensional random vector $\underline{X}$ with support $\mathcal{S}$ is lower–upper counter-monotonic with threshold $\underline{a}$ if the following conditions hold:

1. $\mathcal{S}_{\underline{a}} = \mathcal{S} \cap TL(\underline{a})$ is a counter-monotonic set,
2. $\mathbb{P}[X_1 < a_1, X_2 > a_2] > 0$,
3. $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TL(\underline{a}) \cup \overline{BR}(\underline{a})\right)\right)$ is empty.

The following lemma provides a link between the four different types of quadrant perfect dependence. A proof of the first statement can be found in Appendix A.7, and similar arguments based on the rotation of the support of (X_1, X_2) can be used to prove the remaining statements.

Lemma 5.1. *Consider the random vector $\underline{X} = (X_1, X_2)$ with support $\mathcal{S}$ which is upper–upper comonotonic with threshold $(a_1, a_2) \in \mathcal{S}$. The following holds:*

- *The random vector $(-X_1, -X_2)$ is lower–lower comonotonic with threshold $(-a_1, -a_2)$.*
- *The random vector $(X_1, -X_2)$ is upper–lower counter-monotonic with threshold $(a_1, -a_2)$.*
- *The random vector $(-X_1, X_2)$ is lower–upper counter-monotonic with threshold $(-a_1, a_2)$.*

The following theorem establishes a relationship between quadrant perfect dependence and functions having a monotone tail. A proof of the first statement can be found in Appendix A.8, together with a proof of how the second statement follows from the first (in combination with Lemmas 3.1 and 5.1). A similar approach can be used to prove the remaining statements.

Theorem 5.1. *Consider the comonotonic random vector (X_1, X_2). Define the function h as follows:*

$$h_\alpha(p) = F_{X_1}^{-1(\alpha)}(p) - F_{X_2}^{-1(\alpha)}(p), \qquad \text{for } p \in (0, 1), \tag{5.1}$$

with $\alpha \in [0, 1]$, and for $\pi \in (0, 1)$, we define the point $\underline{a}$ as follows:

$$\underline{a} = \left(F_{X_2}^{-1(\alpha)}(\pi), h_\alpha(\pi)\right). \tag{5.2}$$

The following holds:

- *$(X_2, X_1 - X_2)$ is upper–upper comonotonic with threshold $\underline{a}$ iff the function h_α has a non-decreasing upper tail with threshold π.*
- *$(X_2, X_1 - X_2)$ is lower–lower comonotonic with threshold $\underline{a}$ iff the function h_α has a non-decreasing lower tail with threshold π.*

- $(X_2, X_1 - X_2)$ is upper–lower counter-monotonic with threshold $\underline{a}$ iff the function h_α has a non-increasing upper tail with threshold π.
- $(X_2, X_1 - X_2)$ is lower–upper counter-monotonic with threshold $\underline{a}$ iff the function h_α has a non-increasing lower tail with threshold π.

Note that the link with quadrant perfect dependence in the above theorem is derived by assuming that X_1 and X_2 are comonotonic. Investigating whether this link is valid for other dependence structures of (X_1, X_2) may deserve a separate contribution.

6. Capital reducers

6.1. General setting

Consider an insurance company facing a random loss whose present value is denoted by R_1. The insurance company is required to hold a certain amount of cash such that the liabilities can be covered in the future with a sufficiently high probability. This capital buffer, or solvency capital, is determined using a risk measure, denoted by φ [12–17].

In order to reduce the risk associated with its liabilities, and hence to decrease the amount of required regulatory capital, the insurance company can add an adequate partial hedge to the initial position R_1. Assume that today is time 0, and let R_2 denote the present value of the future time-T cash flow received by the insurer of a candidate hedger. The asset R_2 is expected to partially offset the losses from the liability R_1, and we assume that it can be acquired at time 0 at a predetermined price $\varrho[R_2]$. Note that R_2 is an income random variable.

The present value of the insurer's liability after the income received from the asset R_2 is taken into account, is denoted by the loss random variable Z:

$$Z = R_1 - R_2.$$

Positive values of Z have to be interpreted as losses for the insurance company. If R_2 is a perfect hedge for R_1, we have that $Z = 0$, almost surely. However, in a realistic situation, R_1 has a complex structure for which a perfect hedge does not exist. This means that Z can still take positive values (i.e. $\mathbb{P}[Z > 0]$ is positive), indicating a loss for the insurance company. Hence, Z is the residual liability of the insurance company, i.e. the part of the losses of R_1 that cannot be countered by gains from R_2. The new position of the company now consists of the residual liability Z and the price $\varrho[R_2]$ paid for acquiring R_2:

$$Z + \varrho[R_2].$$

Intuitively, R_2 is expected to be an adequate partial hedge if it is strongly positively dependent with R_1. We consider in this section the case where the random variables R_1 and R_2 are comonotonic, and we can write:

$$(R_1, R_2) \stackrel{d}{=} \left(F_{R_1}^{-1}(U), F_{R_2}^{-1}(U) \right), \tag{6.1}$$

where U is uniformly distributed over [0, 1]. A formal definition of comonotonicity as well as some associated results and applications can be found in [1,18].

6.2. Capital reducers

An asset R_2 is said to be an adequate risk reducer for the liability R_1 if the residual liability $Z = R_1 - R_2$ is 'more preferable' than the initial liability R_1. The notion of 'more preferable' can be quantified by using either integral stochastic orders [19,20], or relevant risk measures. For studies on ordering of sums of random vectors, we refer to Denuit et al. [21] and Cheung et al. [22]. We focus here on the latter, and we quantify the situation where R_2 is an effective risk reducer for R_1 by using a translation invariant risk measure φ. In particular, R_2 is a risk reducer for R_1 if the following inequality holds:

$$\varphi[Z] + \varrho[R_2] \leq \varphi[R_1]. \tag{6.2}$$

If (6.2) holds, we say that R_2 is a φ-reducer for the liability R_1.

Inequality (6.2) has a meaningful interpretation in terms of solvency capital reduction. Suppose that the amount of solvency capital a company has to hold is determined by using some risk measure φ. Then, the right-hand side of Inequality (6.2) corresponds with the amount of capital the insurance company has to hold to cover losses coming from the initial liability R_1, i.e. the liability the insurance company faces before the risk reducer is applied. The left-hand side of Inequality (6.2) corresponds with the total amount of capital needed to buy the risk reducer and to set up the solvency capital one has to hold to cover the remaining losses after the income from the risk reducer has been taken into account. Therefore, Inequality (6.2) implies that buying the risk reducer R_2 leads to a reduction in necessary capital and is therefore considered to be the preferred strategy.

This idea of risk reducers was introduced in [8] in the convex order sense, where the authors evaluate the effectiveness of a cash flow which is comonotonic with the initial liability.[1] A further generalization of convex risk reducers is studied in [9]. This is equivalent with setting $\varrho[R_2] = \mathbb{E}[R_2]$ in (6.2), and setting φ equal to TVaR_p for all $p \in [0, 1]$.

A definition in terms of TVaR_p for all levels $p \in [0, 1]$ is meaningful since it relates to the variability of the residual risk. Nevertheless, in practice, selecting the risk reducer R_2 using the TVaR for all levels p can be too conservative, and insurance companies subject to VaR-based regulation might rather focus on the amount of capital reduction only. This is particularly relevant since the Solvency II directive prescribes the use of the VaR. Thus, in this section, we work in a simplified setting where we focus on one single level p, instead of all $p \in [0, 1]$. In the sequel, we use the following definitions:

Definition 6.1. The cash flow R_2 is said to be a *VaR-reducer* at the level p for R_1 if

$$\varrho[R_2] \leq VaR_p[R_1] - VaR_p[Z], \tag{6.3}$$

where $p \in [0, 1]$ and ϱ is a premium principle.

Intuitively, Condition (6.3) implies that the price $\varrho[R_2]$ we pay to set up the hedge should not be larger than the capital reduction we can obtain. It is worth noting that if R_2 is a VaR-reducer at a level p as well as for all $q \geq p$, then it is also a TVaR-reducer at the level p. In particular, by definition, R_2 is a VaR-reducer at a level q if:

$$\mathrm{VaR}_q[Z] + \varrho[R_2] \leq \mathrm{VaR}_q[R_1].$$

If this inequality holds for all $q \geq p$, then:

$$\frac{1}{1-p} \int_p^1 \left(\mathrm{VaR}_q[Z] + \varrho[R_2] \right) \mathrm{d}q \leq \frac{1}{1-p} \int_p^1 \mathrm{VaR}_q[R_1] \mathrm{d}q,$$

and hence:

$$\mathrm{TVaR}_q[Z] + \varrho[R_2] \leq \mathrm{TVaR}_q[R_1].$$

That is, R_2 is a TVaR-reducer at a level p.

For $\varrho[R_2] = \mathbb{E}[R_2]$, Cheung et al. [8] show that R_2 is a comonotonic TVaR-reducer if the vector $(R_2, R_1 - R_2)$ is comonotonic (Theorem 1 and Lemma 6). Moreover, they provide examples where the intuition of merging R_1 and $-R_2$ to reduce the variance does not hold. Thus, Condition (6.3) is not trivial. In what follows, we derive sufficient conditions for R_2 to be a VaR-reducer. Note that the setting considered in this section is similar to that leading to the equivalence between quadrant perfect dependence and functions having monotone tails. Thus, the results of this section can be derived in either frameworks.

6.3. Sufficient conditions on the price of VaR-reducers

The random difference $Z = R_1 - R_2$ can be expressed as $Z \overset{d}{=} h(U)$, where:

$$h(p) = F_{R_1}^{-1}(p) - F_{R_2}^{-1}(p),$$

and U is uniformly distributed over the unit interval. Taking into account Theorem 3.4, we find that there are two conditions for R_2 to be a VaR-reducer.

The first condition follows from the case where h has a non-decreasing upper/lower tail with threshold c. We find that R_2 is a VaR-reducer at the level p for R_1 if its price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \mathrm{VaR}_p[R_2], \tag{6.4}$$

for $p > \pi_c$ when h has a non-decreasing upper tail, or for $p \leq 1 - \lambda_c$ when h has a non-decreasing lower tail. Note that when R_1 is a loss random variable, we are mostly interested in large values for p, meaning that a function h with non-decreasing upper tail is more useful.

Condition (6.4) provides a simple rule for determining a VaR-reducer, and it has a straightforward interpretation when combined with the concepts of upper–upper comonotonicity and lower–lower comonotonicity. In particular, we have shown in Theorem 5.1 that upper–upper comonotonicity can be linked to the fact that the function h has a non-decreasing upper tail, whereas lower–lower comonotonicity can be linked to the fact that h has non-decreasing lower tail. Suppose that the losses in R_1 are large, i.e. R_1 takes large positive values. Then comonotonicity of (R_1, R_2) implies that R_2 will be large too. Assume that R_2 is exceeding the comonotonic threshold. Then upper–upper comonotonicity of the vector $(R_2, R_1 - R_2)$ ensures that R_2 will not dominate R_1, which would result in a heavy lower tail. Indeed, the loss $R_1 - R_2$ increases if R_2

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[1] Note that the original definition is equivalently given for counter-monotonic risk reducers, where they consider the sum of random variables instead of their difference.

increases. A similar rationale holds when R_2 is below a certain threshold, where lower–lower comonotonicity comes into play.

The second condition follows from the case where h has a non-increasing upper/lower tail with threshold c and R_1 and R_2 are continuous, we find that R_2 is a VaR-reducer at the level p for R_1 if its price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \mathrm{VaR}_p[R_1] - \mathrm{VaR}_{1-p}[R_1] + \mathrm{VaR}_{1-p}[R_2], \tag{6.5}$$

for $p < 1 - \lambda_c$ when h has a non-increasing upper tail, or for $p \geq \pi_c$ when h has a non-increasing lower tail.

Condition (6.5) represents the opposite situation of Condition (6.4). In particular, we know from Theorem 5.1 that the properties of monotone tail functions leading to Inequality (6.5) correspond to quadrant counter-monotonicity. Thus, the loss $R_1 - R_2$ decreases if R_2 increases for the relevant values of p. This means that the cash flow R_2 dominates the initial position R_1 for those values.

Whereas Condition (6.4) can be satisfied as long as the hedger R_2 has a reasonable price, it is not guaranteed that the price of the hedger would satisfy Condition (6.5). Everything held equal, this latter condition would be more likely satisfied in three situations: when the initial position R_1 has a heavy right tail (i.e. high extreme losses have a sufficiently high likelihood), when the initial position has a light left tail (i.e. very low losses have a sufficiently high likelihood), and when the candidate hedger has a heavy left tail (i.e. very low gains have a sufficiently high likelihood). In these cases, the upper bound is larger, and hence, the price of the hedger is likely to satisfy the condition.

It is worth noting that a direct consequence of the decomposition of quantiles in the tails is that the TVaR can also be decomposed in terms of either univariate TVaR's or LTVaR. In case the function h has a non-decreasing upper/lower tail, the TVaR of Z is equal to the difference of the TVaR's at the level p. In contrast, if the function h has non-increasing upper/lower tail, the TVaR of Z is equal to the difference of the LTVaR's at the level $1 - p$. Note that if the underlying random variables are not continuous, the TVaR's and LTVaR's have to be considered carefully as averages of either the left or right quantiles, depending on the nature of h. Taking into account these decompositions, Conditions (6.4) and (6.5) can be extended to TVaR-reducers in a straightforward way.

In what follows, we further analyze these two conditions under some fairly general distributional assumptions.

6.4. Analysis of the conditions for being a VaR-reducer

6.4.1. Distributional assumptions and monotonic properties

We consider the continuous random variable W defined on the real line, with cdf F_W. The inverse cdf is denoted by F_W^{-1}, with $F_W^{-1}(0) = -\infty$ and $F_W^{-1}(1) = +\infty$. The probability density function (pdf) is denoted by f_W and is assumed to be defined on $\mathbb{R}$. Moreover, W is assumed to be standardized, i.e. $\mathbb{E}[W] = 0$ and $\mathrm{Var}[W] = 1$. We assume in this section that the components of (R_1, R_2) can be written as follows:

$$\log R_i = \mu_i + \sigma_i W, \qquad \text{for } i = 1, 2, \tag{6.6}$$

where μ_i and σ_i are finite, and $\sigma_i > 0$, $i = 1, 2$. The VaR of R_i follows from Theorem 3.1:

$$\mathrm{VaR}_p[R_i] = \exp\left(\mu_i + \sigma_i F_W^{-1}(p)\right), \tag{6.7}$$

for $p \in [0, 1]$ and $i = 1, 2$.

The choice (6.6) includes a large set of distributions. For example, one can assume W to be a standard normal distribution, which implies that R_1 and R_2 follow a lognormal distribution. In order to allow more flexibility, one can also assume W to come from a Lévy distribution. The random variables R_1 and R_2 are then exponential Lévy models. Such models have proven useful to model asset returns; see e.g. [23]. In [24], the authors use the model (6.6), where W is assumed to be a standardized Variance Gamma distribution, to model stylized facts of asset returns such as heavy tails, leptokurtic and skewness. They show that this model is accurate for modeling stock returns, especially long term asset returns; see also [25,26].

The function $h : \mathcal{I} \mapsto \mathcal{J}$ can be written as follows:

$$h(u) = e^{\mu_1 + \sigma_1 F_W^{-1}(u)} - e^{\mu_2 + \sigma_2 F_W^{-1}(u)}, \tag{6.8}$$

where $\mathcal{I} = (0, 1)$ and $\mathcal{J} = (y^{\min}, y^{\max})$, with $y^{\min} = \inf_{u \in \mathcal{I}} h(u)$ and $y^{\max} = \sup_{u \in \mathcal{I}} h(u)$.

The following proposition provides conditions under which h is monotone, and shows that when these conditions are not satisfied, then h always has a monotone tail. We refer to Appendix A.9 for a proof; see also [27] for a related result. Note that we discard the trivial case where $\sigma_1 = \sigma_2$ and $\mu_1 = \mu_2$, which leads to $Z = 0$ almost surely.

Proposition 1. *Consider the function h defined in (6.8) and the comonotonic difference $Z \stackrel{d}{=} h(U)$. For $\sigma_1 \neq \sigma_2$, define $c^\star$, $u^\star$ and $y^\star = h(u^\star)$ as follows:*

$$c^\star = F_W\left(\frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2}\right), \qquad \text{and} \qquad u^\star = F_W\left(\frac{\log\left(\frac{\sigma_1}{\sigma_2}\right) + \mu_1 - \mu_2}{\sigma_2 - \sigma_1}\right). \tag{6.9}$$

1. If $\sigma_1 = \sigma_2$ and $\mu_1 > \mu_2$, we have that $y^{\min} = 0$, $y^{\max} = +\infty$, and h is non-decreasing.
2. If $\sigma_1 = \sigma_2$ and $\mu_2 > \mu_1$, we have that $y^{\min} = -\infty$, $y^{\max} = 0$, and h is non-increasing.
3. If $\sigma_1 > \sigma_2$, we have that $y^{\min} = y^\star < 0$, $y^{\max} = +\infty$, and the function h has a non-decreasing upper tail with threshold $c^\star$.
4. If $\sigma_1 < \sigma_2$, we have that $y^{\min} = -\infty$, $y^{\max} = y^\star > 0$, and the function h has a non-increasing upper tail with threshold $c^\star$.

We conclude from Proposition 1 that there are four possible shapes for the function h, in case the marginal distributions are given by (6.6). When $\sigma_1 = \sigma_2$, the function h is always monotone. When $\sigma_1 \neq \sigma_2$, the function h is either convex or concave. Thus, h can have a monotone upper tail, but cannot have a monotone lower tail.

Fig. 2 displays these four shapes. We assume in this figure that W has a standard normal distribution, i.e. R_1 and R_2 are lognormal distributed. The top two panels correspond to the situations where $\sigma_1 = \sigma_2$, i.e. the function h is always monotone. The bottom two panels illustrate the case where $\sigma_1 \neq \sigma_2$, with h being either convex or concave. Thus, h can have a monotone upper tail, but cannot have a monotone lower tail.

6.4.2. Analysis

Now, we further analyze Conditions (6.4) and (6.5) under the present distributional assumptions. First, notice that because R_1 and R_2 are continuous, the right and left inverses F_Z^{-1+} and F_Z^{-1} are equal. For $h^\star = h(c^\star)$, the uniform distribution of U leads to

$$\pi_{c^\star} = c^\star, \qquad \text{and} \qquad \lambda_{c^\star} = 1 - c^\star.$$

Therefore, we conclude that:

Case 1: If either $\sigma_1 = \sigma_2$ and $\mu_1 > \mu_2$, or $\sigma_1 > \sigma_2$, then the hedger R_2 is a VaR-reducer at the level $p \geq c^\star$ for R_1 provided the price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \text{VaR}_p[R_2]. \tag{6.10}$$

Case 2: If either $\sigma_1 = \sigma_2$ and $\mu_1 < \mu_2$, or $\sigma_1 < \sigma_2$, then the hedger R_2 is a VaR-reducer at the level $p \leq c^\star$ for R_1 provided the price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \text{VaR}_{1-p}[R_2] + \text{VaR}_p[R_1] - \text{VaR}_{1-p}[R_1]. \tag{6.11}$$

We first consider Case 1. If the standard deviation of the log-return of R_2 is smaller than that of the initial position, i.e. $\sigma_2 < \sigma_1$, or if these standard deviations are equal but the expected log-return of R_1 is larger than that of R_2, then any hedger R_2 is a VaR-reducer provided the price $\varrho[R_2]$ is not 'too high'. Indeed, it is reasonable to assume that any realistic price $\varrho[R_2]$ will be lower than the amount of capital required for holding R_2 only. Note that in this case, the comonotonic difference Z is bounded from below by $y^\star$.

When R_2 satisfies the conditions of Case 1, its price must be such that $\varrho[R_2] \leq \text{VaR}_p[R_2]$ for all $p \geq c^\star$. Since VaR is non-decreasing with respect to p, then, taking into account (6.7) and (6.9), we find that a sufficient condition on the price of R_2 under Case 1 is as follows:

$$\varrho[R_2] \leq \text{VaR}_{c^\star}[R_2] = \exp\left(\frac{\sigma_1\mu_2 - \sigma_2\mu_1}{\sigma_1 - \sigma_2}\right).$$

Consider now Case 2, where $\sigma_2 > \sigma_1$. In this case, reducing the risk of the liability R_1 by adding the cash flow R_2 'removes' the upper tail. Indeed, the random variable R_2 partially hedges R_1 and takes away part of the losses (i.e. the positive part) of R_1 and the random variable Z is bounded from above. From Proposition (6.9), one can verify that the maximum of the random variable Z is given by:

$$Z \leq e^{\frac{\mu_1\sigma_2 - \mu_2\sigma_1}{\sigma_2 - \sigma_1}} \left(e^{\frac{\sigma_1}{\sigma_2 - \sigma_1}\log\frac{\sigma_1}{\sigma_2}} - e^{\frac{\sigma_2}{\sigma_2 - \sigma_1}\log\frac{\sigma_1}{\sigma_2}} \right), \tag{6.12}$$

whereas its minimum is $-\infty$. Thus, the losses are bounded but the gains are not. However, for R_2 to be a VaR-reducer, its price must satisfy Inequality (6.11), which is not guaranteed. Therefore, there are some situations where the random variable R_2 is too expensive for the capital reduction it provides, and hence, reducing the liability R_1 by adding R_2 is not optimal.

Finally, note that Conditions (6.10) and (6.11) can be extended in a straightforward way to account for TVaR-reducers, and the above discussion remains relevant in this case. In particular, we have that:

Case 1: If either $\sigma_1 = \sigma_2$ and $\mu_1 > \mu_2$, or $\sigma_1 > \sigma_2$, then the hedger R_2 is a TVaR-reducer at the level $p \geq c^\star$ for R_1 provided the price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \text{TVaR}_p[R_2].$$

Case 2: If either $\sigma_1 = \sigma_2$ and $\mu_1 < \mu_2$, or $\sigma_1 < \sigma_2$, then the hedger R_2 is a TVaR-reducer at the level $p \leq c^\star$ for R_1 provided the price $\varrho[R_2]$ satisfies:

$$\varrho[R_2] \leq \text{LTVaR}_{1-p}[R_2] + \text{TVaR}_p[R_1] - \text{LTVaR}_{1-p}[R_1].$$
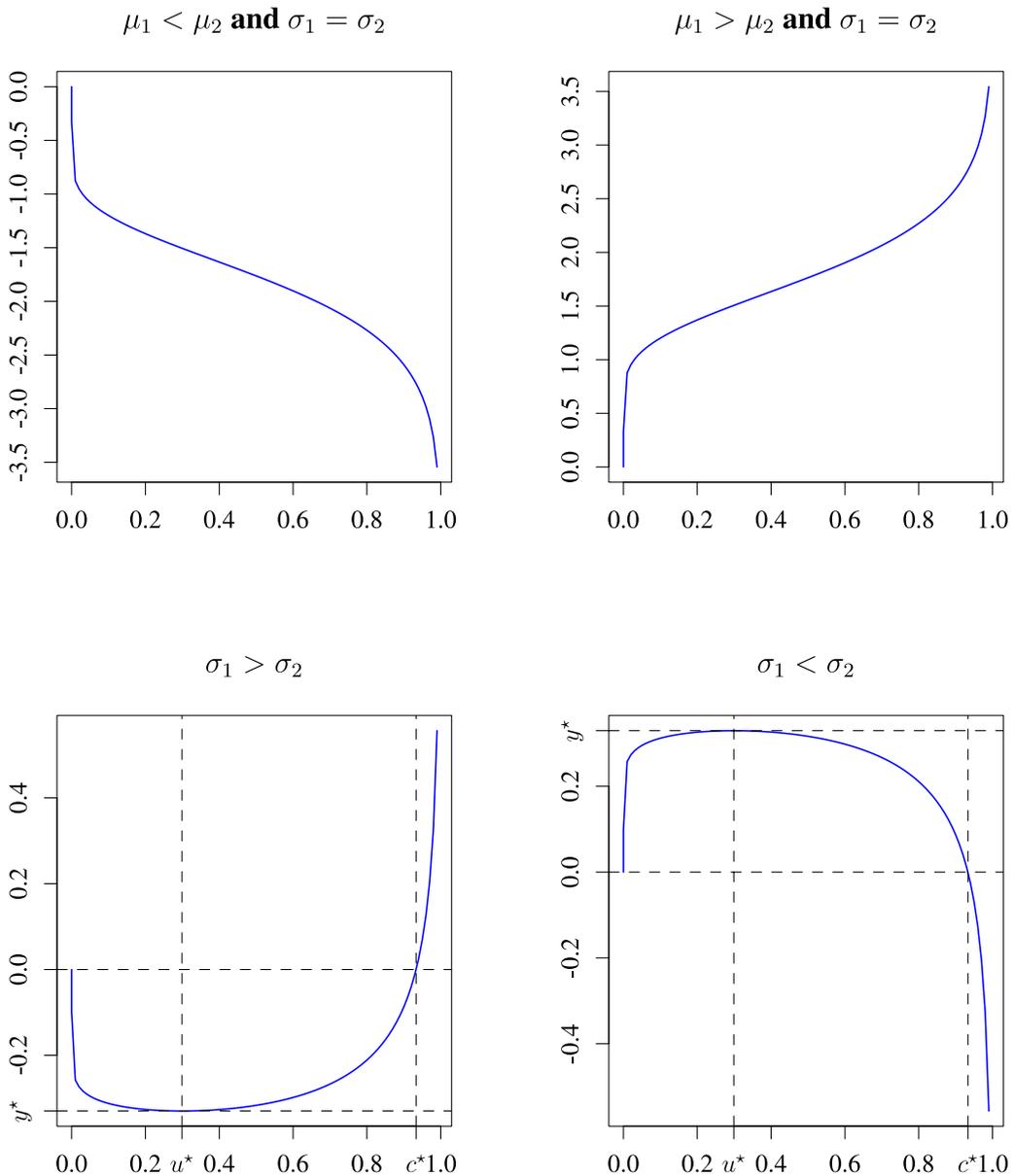
Fig. 2. The function h given by (6.8). The top two panels are the situations where $\sigma_1 = \sigma_2$, whereas the bottom two panels are the situations where $\sigma_1 \neq \sigma_2$. The optimum $u^\star$ and the threshold $c^\star$ are highlighted with the vertical lines.

7. Conclusion

We introduce the notion of monotone tail functions and analyze the properties of random variables transformed using such functions. We extend the results of Dhaene et al. [1] stating that the quantile of a random variable transformed with a monotone function is related to the transformed quantile of the original random variable. In particular, we show that this result holds for monotone tail functions, provided the quantile level is within an explicitly given interval. We also show that these functions are useful in different situations, such as the evaluation of the payoff of option trading strategies, or the present value of endowment assurances.

We address generalizations of the upper comonotonicity structure introduced in [2] through the lens of the proposed theory. We show how quadrant perfect dependence integrates within the notion of monotone tail functions in the framework of risk reducers.

We study the relevance of functions having monotone tails in the context of risk reduction, which is central to many financial and actuarial problems. Often, the problem boils down to selecting a hedger for an initial position, such that the

level of the risk is reduced, and ultimately, the level of required regulatory capital too. We investigate conditions on the price of a comonotonic hedger ensuring a reduction of the Value-at-Risk, and we derive using the theory developed in this paper a simple rule for the choice of the comonotonic hedger.

Appendix. Proofs

A.1. Proof of *Lemma* 3.1

We prove the equivalence between h_u having a non-decreasing upper tail and g_u having a non-increasing upper tail. Similar arguments can be used to prove the remaining statements.

If the function h_u has a non-decreasing upper tail, there exists $x' \in \mathcal{I}$ such that Conditions (1) and (2) in Definition 3.1 hold. On the one hand, Condition (1) states that for $x \in \mathcal{I}$ with $x < x'$, we have that $h_u(x) \leq h(x')$, which is equivalent to $-h_u(x) \geq -h_u(x')$. Thus, if we define the function $g_u : \mathcal{I} \mapsto \tilde{\mathcal{J}}$ such that $g_u(x) = -h_u(x)$, we have that

$$g_u(x) \geq g_u(x'), \qquad \text{for } x \in \mathcal{I} \text{ with } x < x'. \tag{A.1}$$

On the other hand, Condition (2) in Definition 3.1 states that for any $x_1, x_2 \in \mathcal{I}$ with $x' \leq x_1 \leq x_2$, the inequality $h_u(x_1) \leq h_u(x_2)$ holds, which is equivalent with

$$g_u(x_1) \geq g_u(x_2). \tag{A.2}$$

Combining (A.1) and (A.2) proves that g_u satisfies the conditions of a non-increasing upper tail function from Definition 3.2.

Further, we have from (A.1) and (A.2), that any x' satisfying the conditions from Definition 3.1 of non-decreasing upper tail functions (i.e. $x' \in \mathcal{H}_u$), satisfies the conditions from Definition 3.2 of non-increasing upper tail functions. Thus, the infimum of set $\mathcal{H}_u$ associated with h_u is equal to the infimum of set $\mathcal{G}_u$ associated with $-h_u$. Therefore, we conclude that h_u and g_u have the same threshold.

A.2. Proof of *Lemma* 3.2

We can use the law of total probability to write

$$\mathbb{P}\left[\tilde{h}(X) \leq y\right] = \mathbb{P}\left[X < c, \tilde{h}(X) \leq y\right] + \mathbb{P}\left[X = c, \tilde{h}(X) \leq y\right] + \mathbb{P}\left[X > c, \tilde{h}(X) \leq y\right].$$

If $X < c$, then $h(x) \leq h^\star$, and since $h^\star \leq y$, we find

$$\mathbb{P}\left[X < c, \tilde{h}(X) \leq y\right] = \mathbb{P}\left[X < c\right].$$

For the second probability, if we use the definition of $\tilde{h}$ and the fact that $h(c) \leq \max\{h(c), h^\star\}$ to write

$$\mathbb{P}\left[X = c, \tilde{h}(X) \leq y\right] = \mathbb{P}\left[X = c, h(c) \leq y\right].$$

For the third probability, we have by definition that $\tilde{h}(x) = h(x)$ for $x > c$, and thus:

$$\mathbb{P}\left[X > c, \tilde{h}(X) \leq y\right] = \mathbb{P}\left[X > c, h(X) \leq y\right].$$

Combining the alternative representations of these three probabilities and using again the law of total probability, we find that $\mathbb{P}\left[\tilde{h}(X) \leq y\right] = \mathbb{P}\left[h(X) \leq y\right]$.

A.3. Proof of *Lemma* 3.3

In order to prove (3.10), we start with proving the equality

$$F^{-1}_{\tilde{h}(X)}(p) = F^{-1}_{h(X)}(p), \qquad \text{for } p > \pi_c, \tag{A.3}$$

where h has a non-decreasing upper tail. We have from (2.1) that:

$$F^{-1}_{h(X)}(p) = \inf\{y \in \mathbb{R}|\ \mathbb{P}\left[h(X) \leq y\right] \geq p\} \overset{not.}{=} \inf \mathcal{A}_{h(X)},$$

$$F^{-1}_{\tilde{h}(X)}(p) = \inf\left\{y \in \mathbb{R}|\ \mathbb{P}\left[\tilde{h}(X) \leq y\right] \geq p\right\} \overset{not.}{=} \inf \mathcal{A}_{\tilde{h}(X)}.$$

Since $p > \pi_c$, we have for $y \leq h^\star$ that $\mathbb{P}\left[h(X) \leq y\right] < p$ from the definition of π_c, and that $\mathbb{P}\left[\tilde{h}(X) \leq y\right] < p$ from (3.9). This implies that $y \notin \mathcal{A}_{h(X)}$ and $y \notin \mathcal{A}_{\tilde{h}(X)}$ for $y \leq h^\star$, and hence, the sets $\mathcal{A}_{h(X)}$ and $\mathcal{A}_{\tilde{h}(X)}$ contain only values $y > h^\star$. It then follows from (3.8) that $\mathcal{A}_{h(X)} = \mathcal{A}_{\tilde{h}(X)}$, which proves (A.3).

We now prove

$$F_{\tilde{h}(X)}^{-1+}(p) = F_{h(X)}^{-1+}(p), \qquad \text{for } p > \pi_c. \tag{A.4}$$

We use (2.2) to write:

$$F_{h(X)}^{-1+}(p) = \sup \{y \in \mathbb{R}| \; \mathbb{P}\left[h(X) \le y\right] \le p\} \overset{not.}{=} \sup \mathcal{B}_{h(X)},$$

$$F_{\tilde{h}(X)}^{-1+}(p) = \sup \left\{y \in \mathbb{R}| \; \mathbb{P}\left[\tilde{h}(X) \le y\right] \le p\right\} \overset{not.}{=} \sup \mathcal{B}_{\tilde{h}(X)}.$$

Since $p > \pi_c$, we conclude that $h^\star \in \mathcal{B}_{h(X)}$ using the definition of π_c. Similarly, we find that $h^\star \in \mathcal{B}_{\tilde{h}(X)}$ when taking into account (3.9). Thus, $\sup \mathcal{B}_{h(X)} \ge h^\star$ and $\sup \mathcal{B}_{\tilde{h}(X)} \ge h^\star$. The proof of (A.4) is completed by using (3.8) to write:

$$\sup \mathcal{B}_{h(X)} = \sup \left\{y \ge h^\star | \; \mathbb{P}\left[\tilde{h}(X) \le y\right] \le p\right\} = \sup \mathcal{B}_{\tilde{h}(X)}.$$

Finally, combining (A.3) and (A.4) with the definition (2.3) proves (3.10).

A.4. Proof of Lemma 3.4

The inverse F_X^{-1} is defined as follows:

$$F_X^{-1}(p) = \inf \{y \in \mathbb{R}| \; F_X(y) \ge p\} = \inf \mathcal{A}_X.$$

It follows from the definition of $h^\star$ and π_c that we can write:

$$\pi_c = \mathbb{P}\left[X < c\right], \qquad \text{if } h(c) > h^\star, \tag{A.5}$$

and

$$\pi_c \ge \mathbb{P}\left[X \le c\right], \qquad \text{if } h(c) \le h^\star. \tag{A.6}$$

We prove the result for these two cases separately.

Case 1: $h(c) > h^\star$

Since $h(c) > h^\star$, we have that

$$\tilde{h}(c) = \max\left\{h(c), h^\star\right\} = h(c).$$

which implies that:

$$\tilde{h}(x) = h(x), \qquad \text{for } x \ge c. \tag{A.7}$$

Recall from (A.5) that in this case, we can write the probability π_c as follows:

$$\pi_c = \mathbb{P}\left[X < c\right],$$

which leads to:

$$\mathbb{P}\left[X \le c\right] = \pi_c + \mathbb{P}\left[X = c\right]. \tag{A.8}$$

If $\mathbb{P}\left[X = c\right] = 0$, i.e. the cdf F_X does not jump in c and $\pi_c = F_X(c)$, we immediately find that $p > \pi_c$ implies $c \le F_X^{-1}(p)$. If $\mathbb{P}\left[X = c\right] > 0$, i.e. the cdf F_X has a jump in c with $F_X(c) > \pi_c$, then we have either $\pi_c < F_X(c) < p$ or $\pi_c < p \le F_X(c)$. For $\pi_c < F_X(c) < p$, the element p is beyond the jump part corresponding with c, and hence, the monotonicity of F_X^{-1} implies that $c \le F_X^{-1}(p)$. For $\pi_c < p \le F_X(c)$, the element p falls within the jump part, and hence, $c = F_X^{-1}(p)$. Therefore, we have in all cases that:

$$c \le F_X^{-1}(p), \qquad \text{for } p > \pi_c. \tag{A.9}$$

We can use the definition of the α-inverse to conclude that (A.9) implies

$$c \le F_X^{-1}(p) \le F_X^{-1(\alpha)}(p).$$

Taking into account (A.7) proves that $h\left(F_X^{-1(\alpha)}(p)\right) = \tilde{h}\left(F_X^{-1(\alpha)}(p)\right)$ if $p > \pi_c$.

Case 2: $h(c) \leq h^\star$

In case $h(c) \leq h^\star$, we find that the probability π_c can be bounded from below as follows:

$$\pi_c \geq F_X(c), \tag{A.10}$$

which implies that for $p > \pi_c$, we have $p > F_X(c)$. Thus, from the definition of the inverse F_X^{-1}, we find that $c \notin \mathcal{A}_X$. Note that

$$F_X\left(F_X^{-1}(p)\right) \geq p,$$

from which we find that $F_X^{-1}(p) \in \mathcal{A}_X$. Using the non-decreasingness of F_X^{-1} together with the definition of the generalized α-inverse, we have that:

$$c < F_X^{-1}(p) \leq F_X^{-1(\alpha)}(p).$$

Finally, from Expression (3.7) of $\tilde{h}(x)$, we conclude that $h\left(F_X^{-1(\alpha)}(p)\right) = \tilde{h}\left(F_X^{-1(\alpha)}(p)\right)$.

A.5. Proof of Lemma 3.5

For $x < c$, $\tilde{h}(x) = h^\star$ and the function $\tilde{h}$ is continuous. For $x > c$, $\tilde{h}(x) = h(x)$, and the function $\tilde{h}$ has the same continuity as h. It is then sufficient to analyze the functions $\tilde{h}$ and h in c.

Suppose that h is left-continuous. The function h has a non-decreasing upper tail with threshold c, from which we find that $h(c) \leq h^\star$, and hence $\tilde{h}(c) = h^\star$. Therefore, $\tilde{h}$ is left-continuous in c.

Suppose that h is right-continuous. We have that $h(c) \geq h^\star$, which implies $\tilde{h}(c) = h(c)$, and therefore, $\tilde{h}(x) = h(x)$ for $x \geq c$. We can conclude that $\tilde{h}$ is right-continuous.

A.6. Proof of Theorem 3.4

Expression (3.12)

If the function h_u is left-continuous with a non-decreasing upper tail, then $\tilde{h}_u$ as defined in (3.7) is left-continuous and non-decreasing. Using (3.1) gives:

$$F_{\tilde{h}_u(X)}^{-1}(p) = \tilde{h}_u\left(F_X^{-1}(p)\right), \qquad \text{for } p \in (0, 1). \tag{A.11}$$

Using Lemma 3.3 with $\alpha = 1$ gives

$$F_{\tilde{h}_u(X)}^{-1}(p) = F_{h_u(X)}^{-1}(p), \qquad \text{for } \pi_{c_u} < p, \tag{A.12}$$

with $\pi_{c_u} = \mathbb{P}\left[h_u(X) \leq h^\star\right]$ and $h^\star = \sup_{x < c_u} h_u(x)$. If we put $\alpha = 0$ in Lemma 3.4, we find

$$\tilde{h}_u\left(F_X^{-1}(p)\right) = h_u\left(F_X^{-1}(p)\right), \qquad \text{for } \pi_{c_u} < p. \tag{A.13}$$

Plugging Expressions (A.12) and (A.13) in (A.11) proves Expression (3.12).

Expression (3.13)

In order to prove Expression (3.13), we start by noting that $\tilde{h}_u$ is now right-continuous and non-decreasing. Using Expression (3.2) gives

$$F_{\tilde{h}_u(X)}^{-1+}(p) = \tilde{h}_u\left(F_X^{-1+}(p)\right), \qquad \text{for } p \in (0, 1). \tag{A.14}$$

The proof for $p > \pi_{c_u}$ follows from Lemmas 3.3 and 3.4, with $\alpha = 0$.

Let us now consider the case where h_u is right-continuous with $p = \pi_{c_u}$. First, note that the right-inverse $F_{h_u(X)}^{-1+}$ is a right-continuous function, and hence:

$$F_{h_u(X)}^{-1+}(\pi_{c_u}) = \lim_{p \downarrow \pi_{c_u}} F_{h_u(X)}^{-1+}(p).$$

Using the equality in (3.13) for the case $p > \pi_{c_u}$, we find:

$$F_{h_u(X)}^{-1+}(\pi_{c_u}) = \lim_{p \downarrow \pi_{c_u}} h_u\left(F_X^{-1+}(p)\right).$$

Since F_X^{-1+} is a non-decreasing function, the r.h.s. of this expression can be rewritten as

$$\lim_{p \downarrow \pi_{c_u}} h_u\left(F_X^{-1+}(p)\right) = \lim_{x \downarrow F_X^{-1+}(\pi_{c_u})} h_u(x).$$

Using the right continuity of the function h_u, leads to:

$$F_{h_u(X)}^{-1+}(\pi_{c_u}) = h_u\left(F_X^{-1+}(\pi_{c_u})\right),$$

which therefore proves the result for $p \geq \pi_{c_u}$.

Expressions (3.14) *and* (3.15)

We know from Lemma 3.1 that h_u having a non-decreasing upper tail with threshold c_u is equivalent with $g_u = -h_u$ having a non-increasing upper tail with threshold c_u. Moreover, the left or right continuity of h_u is the same as that of g_u. We also have that $g_u^\star = -h_u^\star$, which leads to $\mathbb{P}\left[h_u(X) \leq h_u^\star\right] = \mathbb{P}\left[g_u(X) \geq g_u^\star\right]$.

Therefore, since $g_u = -h_u$, we find that Expression (3.12) is equivalent with:

$$F_{-g_u(X)}^{-1}(p) = -g_u\left(F_X^{-1}(p)\right), \qquad \text{for } p > \lambda_{c_u},$$

and using (3.3) (or, equivalently, from (3.4)), we have that $F_{-g_u(X)}^{-1}(p) = -F_{g_u(X)}^{-1+}(1-p)$, and rearranging for $1-p$ proves (3.14).

Further, Expression (3.13) is equivalent with:

$$F_{-g_u(X)}^{-1+}(p) = -g_u\left(F_X^{-1+}(p)\right), \qquad \text{for } p \geq \lambda_{c_u},$$

and using again (3.3) and rearranging for $1-p$ proves (3.15).

Finally, using the notation for general functions g_u with a non-increasing upper tail with threshold d_u ends the proof.

Expressions (3.16)–(3.19)

The proof is similar to that of (3.14) and (3.15), where we can use Lemma 3.1.

A.7. Proof of Lemma 5.1

We prove the result for

(X_1, X_2) upper–upper comonotonic $\iff (-X_1, -X_2)$ lower–lower comonotonic,

and similar arguments can be used to prove the other results.

Let $\underline{X} = (X_1, X_2)$ be an upper–upper comonotonic random vector with threshold (a_1, a_2), and we denote its support by $\mathcal{S}$. We consider the random vector $\underline{\bar{X}} = (-X_1, -X_2)$ with support $\bar{\mathcal{S}}$. For any $(x_1, x_2) \in \mathcal{S}$, we have that $(-x_1, -x_2) \in \bar{\mathcal{S}}$, which leads to:

$(x_1, x_2) \in \mathcal{S} \cap TR((a_1, a_2)) \iff (-x_1, -x_2) \in \bar{\mathcal{S}} \cap BL((-a_1, -a_2)),$

and therefore:

$\mathcal{S} \cap TR((a_1, a_2))$ is a comonotonic set $\iff \bar{\mathcal{S}} \cap BL((-a_1, -a_2))$ is a comonotonic set.

The proof of the equivalence between the second conditions for lower–lower and upper–upper comonotonicity follows directly as:

$$\mathbb{P}\left[X_1 < a_1, X_2 < a_2\right] > 0 \Leftrightarrow \mathbb{P}\left[\bar{X}_1 > -a_1, \bar{X}_2 > -a_2\right] > 0.$$

For the third condition, it is also straightforward to show that:

$$\bar{\mathcal{S}} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right) \text{ is empty } \iff \bar{\mathcal{S}} \cap \left(\mathbb{R}^2 \setminus \left(BL(-\underline{a}) \cup \overline{TR}(-\underline{a})\right)\right) \text{ is empty}$$

Combining the equivalences between the three conditions ends the proof.

A.8. Proof of Theorem 5.1

First statement: proof of $(\Rightarrow)$

Suppose that $(X_2, X_1 - X_2)$ is upper–upper comonotonic with threshold $\underline{a}$ given by (5.2) with $\alpha \in [0, 1]$, and consider the function h_α defined in (5.1). We want to show that the function h_α has a non-decreasing upper tail with threshold π. From the definition of non-decreasing upper tail functions, we need to show that $h_\alpha(p) \leq h_\alpha(\pi)$ for $p < \pi$, and that $h_\alpha(p_1) \leq h_\alpha(p_2)$, for $\pi \leq p_1 \leq p_2$.

Notice that because (X_1, X_2) is comonotonic, we have that $(X_1, X_2) \stackrel{d}{=} \left(F_{X_1}^{-1(\alpha)}(U), F_{X_2}^{-1(\alpha)}(U)\right)$, where U is uniformly distributed over $[0, 1]$. Hence, $\mathbb{P}\left[(X_2, X_1 - X_2) \in \mathcal{S}\right] = 1$, with:

$$\mathcal{S} = \left\{\left(F_{X_2}^{-1(\alpha)}(p), h_\alpha(p)\right) | p \in [0, 1]\right\}. \tag{A.15}$$

Moreover, since $(X_2, X_1 - X_2)$ is upper–upper comonotonic, then $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right)$ is empty, i.e. the support $\mathcal{S}$ takes values only in the top-right quadrant $TR(\underline{a})$ and the bottom-left quadrant $\overline{BL}(\underline{a})$.

We now prove the first property, i.e. $h_\alpha(p) \leq h_\alpha(\pi)$ for $p < \pi$. Let $p < \pi$ and consider the element $\left(F_{X_2}^{-1(\alpha)}(p), h_\alpha(p)\right) \in \mathcal{S}$. Since quantiles are non-decreasing, we have that $F_{X_2}^{-1(\alpha)}(p) \leq F_{X_2}^{-1(\alpha)}(\pi)$. Moreover, $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right)$ is empty, which means that the element $\left(F_{X_2}^{-1(\alpha)}(p), h_\alpha(p)\right)$ is necessarily in the bottom-left quadrant, and hence:

$$h_\alpha(p) \leq h_\alpha(\pi), \qquad \text{for } p < \pi. \tag{A.16}$$

We now prove the second property, i.e. $h_\alpha(p_1) \leq h_\alpha(p_2)$, for $\pi \leq p_1 \leq p_2$. Consider the elements p_1 and p_2 with $\pi \leq p_1 \leq p_2$. Because $F_{X_2}^{-1(\alpha)}$ is non-decreasing, we have that $F_{X_2}^{-1(\alpha)}(p_j) \geq F_{X_2}^{-1(\alpha)}(\pi)$ for $j = 1, 2$. Note that the top-right quadrant $TR(\underline{a})$ does not contain the element $\underline{a}$. Thus, we distinguish two cases. In the first case where $F_{X_2}^{-1(\alpha)}(p_j) = F_{X_2}^{-1(\alpha)}(\pi)$, we have that $h_\alpha(p_j) = F_{X_1}^{-1(\alpha)}(p_j) - F_{X_2}^{-1(\alpha)}(\pi)$, and by the monotonicity of $F_{X_1}^{-1(\alpha)}$, we find that $h_\alpha(p_j) \geq h_\alpha(\pi)$. In the second case where $F_{X_2}^{-1(\alpha)}(p_j) > F_{X_2}^{-1(\alpha)}(\pi)$, we use again the fact that $\left(F_{X_2}^{-1(\alpha)}(p_j), h_\alpha(p_j)\right)$ is necessarily in the top-right quadrant, and hence, $h_\alpha(p_j) > h_\alpha(\pi)$ for $j = 1, 2$ also holds. Thus, we have that $h_\alpha(p_j) \geq h_\alpha(\pi)$ for $j = 1, 2$ and $\pi \leq p_1 \leq p_2$.

Invoking again the non-decreasingness of $F_{X_2}^{-1(\alpha)}$, together with the monotonicity of the set $\mathcal{S}_{\underline{a}} = \mathcal{S} \cap TR(\underline{a})$, leads to:

$$h_\alpha(p_1) \leq h_\alpha(p_2), \qquad \text{for } \pi \leq p_1 \leq p_2. \tag{A.17}$$

Inequalities (A.16) and (A.17) imply that h_α has a non-decreasing upper tail with threshold π.

First statement: proof of ($\Leftarrow$)

Suppose the function h_α defined in (5.1) has a non-decreasing upper tail with threshold $\pi \in (0, 1)$. We aim at showing that the vector $(X_2, X_1 - X_2)$ is upper–upper comonotonic with threshold $\underline{a}$ given in (5.2). For this, we need to show that the three conditions for upper–upper comonotonicity given in Definition 5.1 are satisfied. Namely, that $\mathcal{S}_{\underline{a}}$ is a comonotonic set, that $\mathbb{P}[\underline{X} > \underline{a}] > 0$, and that the set $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right)$ is empty.

Recall that the support of $(X_2, X_1 - X_2)$ is given by the set $\mathcal{S}$ defined in (A.15). Moreover, the point $\underline{a} = \left(F_{X_2}^{-1(\alpha)}(\pi), h_\alpha(\pi)\right)$ is an element of $\mathcal{S}$.

First, the set $\mathcal{S}_{\underline{a}}$ is given by:

$$\mathcal{S}_{\underline{a}} = \mathcal{S} \cap TR(\underline{a}) = \left\{\left(F_{X_2}^{-1(\alpha)}(p), h_\alpha(p)\right) | p > \pi\right\}.$$

The quantile $F_{X_2}^{-1(\alpha)}$ is non-decreasing for $p \in [0, 1]$, and h has a non-decreasing upper tail with threshold π, and hence, h_α is non-decreasing for $p \geq \pi$. Thus, $\mathcal{S}_{\underline{a}}$ is a comonotonic set.

Second, for $\pi < 1$, the set $\mathcal{S}_{\underline{a}}$ is not empty, and hence $\mathbb{P}[(X_2, X_1 - X_2) > \underline{a}] > 0$.

Third, we combine again the fact that the quantile $F_{X_2}^{-1(\alpha)}$ is non-decreasing and that h_α has a non-decreasing upper tail with threshold π to conclude that, on the one hand, we have $h_\alpha(p) \geq h_\alpha(\pi)$ and $F_{X_2}^{-1(\alpha)}(p) \geq F_{X_2}^{-1(\alpha)}(\pi)$ for $p \geq \pi$, and on the other hand, we have $h_\alpha(p) \leq h_\alpha(\pi)$ and $F_{X_2}^{-1(\alpha)}(p) \leq F_{X_2}^{-1(\alpha)}(\pi)$ for $p < \pi$. Therefore, the support $\mathcal{S}$ of $(X_2, X_1 - X_2)$ takes values in either the top-right or the bottom-left quadrants. We conclude that the set $\mathcal{S} \cap \left(\mathbb{R}^2 \setminus \left(TR(\underline{a}) \cup \overline{BL}(\underline{a})\right)\right)$ is empty, which proves the result.

Second statement

For $\pi \in (0, 1)$ and $\alpha \in [0, 1]$, let $(X_2, X_1 - X_2)$ be lower–lower comonotonic with threshold

$$\underline{a} = (a_1, a_2) = \left(F_{X_2}^{-1(\alpha)}(\pi), F_{X_1}^{-1(\alpha)}(\pi) - F_{X_2}^{-1(\alpha)}(\pi)\right),$$

Lemma 5.1 states that the random vector $\left(\bar{R}_2, \bar{R}_1 - \bar{R}_2\right)$, where $\bar{R}_i = -X_i$, is upper–upper comonotonic with threshold

$$\bar{\underline{a}} = (-a_1, -a_2) = \left(F_{\bar{R}_2}^{-1(1-\alpha)}(1 - \pi), F_{\bar{R}_1}^{-1(1-\alpha)}(1 - \pi) - F_{\bar{R}_2}^{-1(1-\alpha)}(1 - \pi)\right),$$

where we use the fact that $-F_{X_i}^{-1(\alpha)}(\pi) = F_{\bar{R}_i}^{-1(1-\alpha)}(1 - \pi)$. We define the function $\bar{h}_\alpha$ as follows:

$$\bar{h}_\alpha(p) = F_{\bar{R}_1}^{-1(\alpha)}(p) - F_{\bar{R}_2}^{-1(\alpha)}(p), \qquad \text{for } p \in (0, 1).$$

We can write:

$$\bar{\underline{a}} = \left(F_{\bar{R}_2}^{-1(1-\alpha)}(1 - \pi), \bar{h}_{1-\alpha}(1 - \pi)\right).$$

Since (X_1, X_2) is comonotonic, we also have that $\left(\bar{R}_1, \bar{R}_2\right)$ is comonotonic. Thus, from Theorem 5.1, the upper–upper comonotonicity of $(X_2, X_1 - X_2)$ is equivalent with $\bar{h}_{1-\alpha}$ having a non-decreasing upper tail with threshold $1 - \pi$. Define the function h_α as follows:

$$h_\alpha(p) = F_{X_1}^{-1(\alpha)}(p) - F_{X_2}^{-1(\alpha)}(p).$$

We have that $h_\alpha(p) = -\bar{h}_{1-\alpha}(1 - p)$ for $p \in (0, 1)$. Using Lemma 3.1, the function $-\bar{h}_{1-\alpha}(-p)$, for $p \in (-1, 0)$, has a non-decreasing lower tail with threshold $\pi - 1$. A change of variable leads to the conclusion that $h(p) = -\bar{h}_{1-\alpha}(1 - p)$, for $p \in (0, 1)$, has a non-decreasing lower tail with threshold π, which proves the result.

A.9. Proof of Proposition 1

We start with the monotonicity of the function h. The cdf F_W is strictly increasing and continuous, and hence, its inverse F_W^{-1} is differentiable. This implies that the function h defined in (6.8) is also differentiable. We have that

$$\frac{dF_W^{-1}(u)}{du} = \frac{1}{f_W\left(F_W^{-1}(u)\right)} > 0,$$

which leads to the following expression for the derivative of h:

$$\frac{dh}{du} = \frac{1}{f_W\left(F_W^{-1}(u)\right)} \left(\sigma_1 e^{\mu_1 + \sigma_1 F_W^{-1}(u)} - \sigma_2 e^{\mu_2 + \sigma_2 F_W^{-1}(u)}\right).$$

Studying the sign of this derivative is equivalent to studying the sign of

$$\log \frac{\sigma_1}{\sigma_2} + \mu_1 - \mu_2 - (\sigma_2 - \sigma_1) F_W^{-1}(u).$$

If $\sigma_1 = \sigma_2$, the function h is always monotone, and the sign of $\frac{dh}{du}$ depends on the sign of $\mu_1 - \mu_2$. If $\sigma_1 < \sigma_2$, we find that $\frac{dh}{du}$ is positive on $[0, u^\star]$ and negative on $[u^\star, 1]$, where

$$u^\star = F_W\left(\frac{\log \frac{\sigma_1}{\sigma_2} + \mu_1 - \mu_2}{\sigma_2 - \sigma_1}\right). \tag{A.18}$$

Hence, the function h starts at zero, is non-decreasing until it reaches its maximum in $u^\star$, after which the function h is non-increasing and tends to $-\infty$. Analogously, if $\sigma_1 > \sigma_2$, we find that the function h starts at zero and is non-increasing in $[0, u^\star]$. In the interval $[u^\star, 1]$, the function is non-decreasing and tends to $+\infty$. Also, we find that for $\sigma_1 \neq \sigma_2$, the function h has a monotone upper tail with threshold $c^\star$ such that $h(c^\star) = 0$, and hence, $c^\star = F_W\left(\frac{\mu_2 - \mu_1}{\sigma_1 - \sigma_2}\right)$.

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