Value-at-Risk, Tail Value-at-Risk and upper tail transform of the sum of two counter-monotonic random variables

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Abstract

It is well-known that the Value-at-Risk (VaR) of comonotonic sums can be decomposed into marginal VaR's at the same level. This additivity property allows to derive useful decompositions for other risk measures. In particular, the Tail Value-at-Risk (TVaR) and the upper tail transform of comonotonic sums can be written as the sum of their corresponding marginal risk measures.

The other extreme situation, involving the sum of two arbitrary counter-monotonic random variables, presents a certain number of challenges. One of them is that it is not straightforward to express the VaR of a counter-monotonic sum in terms of the VaR's of the marginal components of the sum. This paper contributes to the literature by proposing solutions to overcome those challenges. The decompositions of the VaR, TVaR and the stop-loss transform of the sum of two arbitrary random variables are derived in terms of their corresponding marginal risk measures.

Keywords: Counter-monotonicity, risk aggregation, risk measures.

1 Introduction

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

$$1 + i = \left(1 + \frac{i^{(2)}}{2}\right)^2$$

$$\frac{i^{(2)}}{2} = 12\%$$

$$i = 1.12^2 - 1$$

This paper provides decompositions for the Value-at-Risk (VaR), Tail Value-at-Risk (TVaR), and the upper tail transform (or stop-loss premium), of the sum of two counter-monotonic random variables with arbitrary marginal distributions. It is shown that the VaR can be expressed in terms of marginal quantiles but at different levels. Further, it is shown that the TVaR and the upper tail transform can be expressed in terms of linear combinations of the marginal risk measures, with additional terms required when the underlying random variables are discrete. The uniqueness of the representation of the VaR and the TVaR in terms of marginal risk measures is however not guaranteed in general.

The results of the present paper are useful in situations involving sums or differences of two random variables. These situations underlie a broad range of practical actuarial and financial problems. In an insurance context, asset and liability management requires the analysis of a difference of two random variables. The liability side of the balance sheet is in major part determined by actuarial risks, whereas the asset side, on the other hand, is a mirror image of the insurer's investment strategy (Wüthrich, 2020). Other situations include the evaluation of basket and spread options, whose payoffs are expressed as stop-loss premiums of sums, or differences, of random variables (Dhaene and Goovaerts, 1996, Carmona and Durrleman, 2003, Laurence and Wang, 2008, 2009). The complex instruments from the emerging market of longevity and mortality derivatives, such as longevity trend bonds or catastrophic mortality bonds, also involve differences of random variables (Hunt and Blake, 2015, Chen et al., 2015a, Bahl and Sabanis, 2021), and so does the management of basis risk (Coughlan et al., 2011, Zhang et al., 2017).

Determining the distribution and risk measures for linear combinations of random variables requires modeling their joint distribution. As pointed out in a number of earlier related studies (see e.g. Kaas et al. (2000), Bernard and Vanduffel (2015), Bignozzi et al. (2015)), modeling

the individual risks, or marginal distributions, is often considered a standard task by actuaries and risk managers, whereas a more challenging task consists in determining their dependence structure. Indeed, an extensive statistical and quantitative toolbox is available to model separately actuarial risks, such as future mortality dynamics or claim counts in non-life insurance, and financial risks, such as future returns of an investment portfolio. However, the joint behavior of actuarial and financial risks is hard to estimate, and it is not uncommon that the problem is simplified by assuming an independence structure.

A reasonable risk management strategy consists in analyzing the most extreme situations for the joint behavior of a random vector with given marginal distributions. For sums of two random variables, these extreme situations are modeled by comonotonicity (i.e. extreme positive dependence) and counter-monotonicity (i.e. extreme negative dependence). In the framework of convex ordering, comonotonicity corresponds to the worst-case scenario and counter-monotonicity corresponds to the best-case scenario (Denuit et al., 2005, Shaked and Shanthikumar, 2007, Cheung and Lo, 2014). Therefore, the TVaR and the upper tail transform of the random sum will reach their maximal and minimal value in case the random variables are comonotonic and counter-monotonic, respectively. Under the assumption of extreme dependence structures, the TVaR and the upper tail transform can be used to assess the diversification benefit from combining different risks or investment opportunities (Embrechts et al., 2009, 2013). Further, since even the most hardened modelers are not exempt from model risk, it is prudent to use sharp comonotonic and counter-monotonic bounds which are consistent with the available information on the univariate distributions (Dhaene et al., 2000, Kaas et al., 2000). These bounds are also informative on the extent of dependence model risk, which can be quantified using the dependence uncertainty spread, i.e. the difference between the upper and lower bounds (Embrechts et al., 2015, Bignozzi et al., 2015). In particular, the knowledge about the marginal distributions does not unambiguously determine the value of a risk measure, since infinitely many values can be obtained from the spectrum of all possible dependence structures. Hence, the dependence uncertainty spread allows insurers and risk managers to evaluate their vulnerability to the choice of a dependence structure[1].

The decompositions of the TVaR and the stop-loss premium of comonotonic sums are standard

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[1]A substantial amount of research has been devoted to derive alternative analytical and numerical worst- and best-case bounds. For the VaR, these bounds are necessary because comonotonic and counter-monotonic VaR's are not always worst- and best-case bounds. Some notable contributions in this direction are due to Mesfioui and Quessy (2005), Wang et al. (2013), Embrechts et al. (2013), Bernard et al. (2017), and more recently Luxa and Papapantoleon (2019). Approaches tailored for the TVaR or for distortion risk measures that incorporate additional information can be found in Kaas et al. (2000), Puccetti (2013) and Bernard et al. (2014). Nevertheless, a trade-off arises from this discussion. Including more information would indeed tighten the bounds, but it could also add to the dependence model risk, because of the uncertainty that stems from that information. In contrast, the comonotonic and counter-monotonic bounds provide more safety in the analysis of the dependence uncertainty spread, at the cost of a wider interval.

results in actuarial science (Meileison and Nádas, 1979, Dhaene et al., 2000, 2006, Hobson et al., 2005, Chen et al., 2008). A key property underpinning these results is the additivity of quantiles of comonotonic sums, which is itself due to the non-decreasing properties of quantiles (Dhaene et al., 2002a, McNeil et al., 2015). Solving the corresponding problem involving extreme negative dependence is more challenging. In the bivariate case[2], which is of interest to the present paper, deriving explicit decomposition formulas for the VaR, TVaR, and stop-loss premium of the sum of counter-monotonic random variables with arbitrary marginal distributions remains an open problem.

For the equivalent problem of comonotonic differences, Laurence and Wang (2009) derive a decomposition for stop-loss premiums in the case of two continuous random variables, but they do not provide closed-form expressions for the corresponding retentions. Chaoubi et al. (2020) also restrict their analysis to continuous random variables with a specific choice of marginal distributions, including symmetric or unimodal distributions. When the random variables follow such marginal distributions, the function defined as the sum of quantiles of counter-monotonic random variables have appealing properties. In particular, it is continuous, and either strictly monotone or globally convex/concave. However, this paper shows that many situations involve functions with complex shapes which are not covered in the existing literature. A typical case is when one of the random variables is discrete (e.g. number of survivors) whereas the other is continuous (e.g. investment fund value). Another one is when the marginal distributions have fat tails or are skewed. In such situations, the existing decompositions are not valid. Hanbali and Linders (2021) provide decompositions for the VaR of comonotonic differences under more general assumptions using the concept of tail monotone functions, where the decomposition holds for certain values of the quantile level. Their results can readily be applied to decompose the TVaR as well, but only at confidence levels beyond some thresholds.

The present paper contributes to the body of literature devoted to the decomposition of risk measures of sums of two random variables (Dhaene and Denuit, 1999, Cheung and Lo, 2013, Cheung et al., 2017). The paper extends the work of Laurence and Wang (2009) and Chaoubi

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[2]Characterizing extreme negative dependence for more than two random variables is still an active area of research. One of the proposed notions is complete mixability, which minimizes the variance of sums of random variables and coincides with counter-monotonicity in the bivariate case; see Gaffke and Rüschendorf (1981) for the foundational principles and Wang and Wang (2011), Wang et al. (2013) and references therein for additional results pertaining to VaR minimization. Dhaene and Denuit (1999) proposed mutual exclusivity, which leads to convex lower bounds for some classes of multivariate distributions with given marginals. Cheung and Lo (2014) provide details on the intrinsic relationship between pairwise counter-monotonicity and multivariate mutual exclusivity. Other proposals for multivariate extreme negative dependence include Lee and Ahn (2014)'s d-counter-monotonicity and Puccetti and Wang (2015)'s Σ-counter-monotonicity; see also Wang and Wang (2015, 2016). Nevertheless, this fertile literature has scarcely addressed the decomposition of the TVaR and the stop-loss premiums. A noteworthy result is due to Dhaene and Denuit (1999) who proved the additivity of stop-loss premiums for sums of mutually exclusive risks under some conditions on the marginal distributions; see also Cheung and Lo (2013) for other convex functionals of sums, as well as Cheung et al. (2017) for tail mutual exclusivity.

et al. (2020) by deriving explicit decompositions of the VaR, TVaR and stop-loss premium of counter-monotonic sums with general marginal distributions. Whereas the decomposition of the VaR is a natural result that follows from the basic definitions, the decomposition of TVaR's and stop-loss premiums are more challenging. This is due to the fact that the sum of quantiles of two counter-monotonic random variables may exhibit complex features, such as jumps (and hence, does not fall in the framework considered in Laurence and Wang (2009) and Chaoubi et al. (2020) with continuous random variables) or multiple inflection points (and hence, for VaR and TVaR, violates Chaoubi et al. (2020)'s convex/concave assumptions).

A standard result in the literature is that the quantile of a random variable transformed by a monotone function can be expressed as the quantile of the original function transformed using that function (Dhaene et al., 2002a). For instance, for a continuous and non-decreasing function g and a random variable X, it holds that $F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p)\right)$ for all $p \in (0, 1)$. The situation where g is not monotone is problematic. Hanbali and Linders (2021) address the problem and show that the equality $F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p)\right)$ holds for specific values of p if the function g has a monotone tail. The present paper adds to their insights by showing that if g is not monotone, then that equality does not necessarily hold. Further, there does not always exist a unique $p^\star$ such that $F_{g(X)}^{-1}(p) = g\left(F_X^{-1}(p^\star)\right)$.

The remainder of the paper is organized as follows. Section 2 introduces the notations and relevant definitions. A definition of the companion comonotonicity concept is also provided, as well as the analogous results available in the literature. Section 3 contains a formal definition of counter-monotonicity. The main problem of the paper is exposed, and some key elements for the subsequent derivations are provided, together with some numerical examples. The main results of the paper are derived in Sections 4, 5 and 6, dealing with the decompositions of the VaR, TVaR and the stop-loss premium, respectively. Each section contains the main results, followed by a discussion and numerical illustrations. Section 7 concludes the paper. In order to ease the reading, all proofs are relegated to the appendix.

2 Preliminaries

2.1 Notations

All random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. They can be discrete, continuous, or a combination of the two. It is assumed that all random variables are such that the risk measures introduced hereafter are finite.

The cumulative distribution function (cdf) of a random variable X is denoted by F_X. The left inverse of F_X is denoted by F_X^{-1}, and is defined as follows:

$$F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \qquad p \in [0,1],$$

with $\inf \emptyset = +\infty$ by convention. The Value-at-Risk (VaR) of a random variable X at the level $p \in [0,1]$ is given by:

$$\text{VaR}_p[X] = F_X^{-1}(p).$$

The right inverse of F_X is denoted by F_X^{-1+}, and is defined as follows:

$$F_X^{-1+}(p) = \sup\{x \in \mathbb{R} \mid F_X(x) \leq p\}, \qquad p \in [0,1],$$

with $\sup \emptyset = -\infty$. If F_X is continuous, the left and right inverses are equal. Otherwise, on horizontal segments of F_X, the inverses $F_X^{-1}(p)$ and $F_X^{-1+}(p)$ are different. In such cases, the generalized α-inverse $F_X^{-1(\alpha)}$, for any $\alpha \in [0,1]$, is defined as follows:

$$F_X^{-1(\alpha)}(p) = (1-\alpha)F_X^{-1}(p) + \alpha F_X^{-1+}(p), \qquad p \in [0,1]. \tag{2.1}$$

More details on generalized inverses can be found in Dhaene et al. (2002a) and Embrechts and Hofert (2013), among others.

The Tail-Value-at-Risk (TVaR) at the level $p \in [0,1]$ of a random variable X is defined as follows:

$$\text{TVaR}_p[X] = \frac{1}{1-p}\int_p^1 F_X^{-1}(q)\mathrm{d}q, \tag{2.2}$$

and the Lower Tail-Value-at-Risk (LTVaR) at the level $p \in [0,1]$ of the random variable X is defined as follows:

$$\text{LTVaR}_p[X] = \frac{1}{p}\int_0^p F_X^{-1}(q)\mathrm{d}q. \tag{2.3}$$

The upper tail transform of the random variable X at the level $x \in \mathbb{R}$, also called the stop-loss premium with retention x, is denoted by $\pi_X(x)$, and is given by the following expected value:

$$\pi_X(x) = \mathbb{E}\left[(X-x)_+\right] = \int_x^{+\infty}(1-F_X(y))\,\mathrm{d}y, \tag{2.4}$$

where $(x)_+ = \max\{x,0\}$. The lower tail transform of X at the level $x \in \mathbb{R}$ is denoted by $\lambda_X(x)$, and is given by the following expected value:

$$\lambda_X(x) = \mathbb{E}\left[(x-X)_+\right] = \int_{-\infty}^x F_X(y)\mathrm{d}y. \tag{2.5}$$

The upper and lower tail transforms $\pi_X(x)$ and $\lambda_X(x)$ are linked through the following relation:

$$\pi_X(x) - \lambda_X(x) = \mathbb{E}[X] - x, \tag{2.6}$$

which is referred to as the *put-call parity* in quantitative finance.

For any $p \in (0,1)$ and $\alpha \in [0,1]$, the following relation establishes a link between the TVaR and the upper tail transform:

$$\text{TVaR}_p[X] = F_X^{-1(\alpha)}(p) + \frac{1}{1-p}\pi_X\left(F_X^{-1(\alpha)}(p)\right), \tag{2.7}$$

which can be rearranged to obtained the link between the LTVaR and the lower tail transform:

$$\text{LTVaR}_p[X] = F_X^{-1(\alpha)}(p) - \frac{1}{p}\lambda_X\left(F_X^{-1(\alpha)}(p)\right). \tag{2.8}$$

The proof of (2.7) can be found in e.g. Dhaene et al. (2006).

Consider the bivariate vector (X_1, X_2) and the sum S given by:

$$S = X_1 + X_2.$$

The goal is to derive decomposition formulas for the VaR, TVaR, and stop-loss premium of the sum S, where the components of the vector (X_1, X_2) are counter-monotonic, i.e. extreme negative dependence. In order to better understand the challenges when working with counter-monotonic random variables, it is helpful to shortly revisit the analogous results for comonotonicity. This is the focus of the remainder of this section.

2.2 Comonotonic sums

Definition 2.1 (Comonotonic modification) *The* comonotonic modification *of a bivariate random vector $\underline{X}$ is denoted by $\underline{X}^c = (X_1^c, X_2^c)$, and is given by:*

$$\underline{X}^c \overset{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)\right), \tag{2.9}$$

where $\overset{d}{=}$ stands for equality in distribution, and U is uniformly distributed over the unit interval.

Characterization (2.9) shows that the components of the comonotonic random vector $\underline{X}^c$ are jointly driven by a single random source transformed by the non-decreasing functions $F_{X_1}^{-1}$ and $F_{X_2}^{-1}$. The marginals of the comonotonic modification remain, however, equal in distribution

to those of the original vector $\underline{X}$. Comonotonicity was extensively discussed in Dhaene et al. (2002a), and its applications in finance and actuarial science are numerous; see e.g. Dhaene et al. (2002b), Dhaene et al. (2005), Denuit and Dhaene (2007), Deelstra et al. (2011), Feng et al. (2015) and Hanbali and Linders (2019).

Let S^u be the sum of the components of the comonotonic modification $\underline{X}^c$. Then:

$$S^u \stackrel{d}{=} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(U).$$

For any dependence structure of the random vector (X_1, X_2), both the TVaR and the stop-loss premium of the sum $S = X_1 + X_2$ are bounded from above by their comonotonic counterparts:

$$
\begin{aligned}
\text{TVaR}_p[S] &\leq \text{TVaR}_p[S^u], &\text{for all } p \in [0, 1], \\
\pi_S(x) &\leq \pi_{S^u}(x), &\text{for all } x \in \mathbb{R}.
\end{aligned}
\tag{2.10}
$$

The proofs of the inequalities in (2.10) can be found in Dhaene et al. (2000) and Kaas et al. (2000).

Quantiles are non-decreasing functions, and hence, the function $f : u \mapsto F_{X_1}^{-1}(u) + F_{X_2}^{-1}(u)$ is also non-decreasing. It is well-known that the quantile of a random variable transformed by a monotone function can be expressed as the quantile of the original random variable transformed using that function (Dhaene et al., 2002a). Thus, combining $S^u \stackrel{d}{=} f(U)$ and the non-decreasingess of f, the following decomposition of the Value-at-Risk holds:

$$\text{VaR}_p[S^u] = f(p) = \text{VaR}_p[X_1] + \text{VaR}_p[X_2], \qquad \text{for all } p \in [0, 1].$$

Note that this decomposition is also valid when S^u is a sum of more than two random variables; see e.g. Dhaene et al. (2002a). Taking into account (2.2), it holds that:

$$\text{TVaR}_p[S^u] = \text{TVaR}_p[X_1] + \text{TVaR}_p[X_2], \qquad \text{for all } p \in [0, 1]. \tag{2.11}$$

This decomposition is also not limited to the two-dimensional case; see e.g. Dhaene et al. (2006).

For any $x \in \left(F_{S^u}^{-1}(0), F_{S^u}^{-1+}(1)\right)$, the comonotonic stop-loss premium $\pi_{S^u}(x)$ admits the following decomposition:

$$\pi_{S^u}(x) = \pi_{X_1}(x_1) + \pi_{X_2}(x_2),$$

where

$$x_i = F_{X_i}^{-1(\alpha_x)}(F_{S^u}(x)), \tag{2.12}$$

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for $i = 1, 2$, and α_x follows from:

$$x_1 + x_2 = x.$$

The fact that the stop-loss premium of a comonotonic sum can be expressed as a sum of the marginal stop-loss premiums is a long-established result (see Meileison and Nádas (1979)), and is valid regardless of the number of components of the comonotonic sum S^u. The explicit expression (2.12) for the retentions x_i was derived in Dhaene et al. (2000); see also Hobson et al. (2005), Chen et al. (2008), Chen et al. (2015b) and Linders et al. (2012).

3   Counter-monotonic sums

3.1   Definition

Definition 3.1 (Counter-monotonic modification) *The* counter-monotonic modification *of a bivariate random vector $\underline{X}$ is denoted by $\underline{X}^{cm} = (X_1^{cm}, X_2^{cm})$, and is given by:*

$$\underline{X}^{cm} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), F_{X_2}^{-1}(1-U) \right),$$

where $\stackrel{d}{=}$ stands for equality in distribution, and U is uniformly distributed over the unit interval.

The counter-monotonic modification $\underline{X}^{cm}$ of a random vector $\underline{X}$ has the same marginal distributions as $\underline{X}$, but the components of $\underline{X}^{cm}$ move perfectly in the opposite direction. Hence, counter-monotonicity corresponds to extreme negative dependence structure.

Counter-monotonicity leads to a convex lower bound for a sum of dependent risks; see e.g. Rüschendorf (1983), Dhaene and Denuit (1999) and Cheung and Lo (2014). Let S^l be the sum of the components of the transformed vector $\underline{X}^{cm}$, such that:

$$S^l = F_{X_1}^{-1}(U) + F_{X_2}^{-1}(1-U).$$

Analogously to the inequalities in (2.10), for any dependence structure of the random vector (X_1, X_2), both the TVaR and the stop-loss premium of the sum $S = X_1 + X_2$ are bounded from below by their counter-monotonic counterparts:

$$\begin{aligned}
\text{TVaR}_p[S^l] &\leq \text{TVaR}_p[S], && \text{for all } p \in [0, 1], \\
\pi_{S^l}(x) &\leq \pi_S(x), && \text{for all } x \in \mathbb{R}.
\end{aligned}$$

But the analogy with comonotonic sums breaks down at this point. In particular, the decomposition of the VaR, TVaR and stop-loss premium of the counter-monotonic sum of arbitrary

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random variables is still an open problem, to which the present paper is devoted.

In order to understand the challenge, recall the function f:

$$f(u) = F_{X_1}^{-1}(u) + F_{X_2}^{-1}(u), \qquad \text{for } u \in (0,1), \tag{3.1}$$

and consider the following function g:

$$g(u) = F_{X_1}^{-1}(u) + F_{X_2}^{-1}(1-u), \qquad \text{for } u \in (0,1), \tag{3.2}$$

with $S^u \overset{d}{=} f(U)$ and $S^l \overset{d}{=} g(U)$. A crucial property underpinning the decompositions of risk measures of comonotonic sums is that the function f defined in (3.1) is always non-decreasing. In contrast, the function g defined in (3.2) is not necessarily monotone, which complicates the derivations.

For some classes of marginal distributions, the problem of determining decomposition formulas is relatively simple, and the function g is often monotone under some easily identifiable conditions. For instance, this is the case when $\log(X_i) - \mathbb{E}\left[log(X_i)\right]$ for $i = 1, 2$ are symmetric and identically distributed. Another example is when X_1 and X_2 are such that:

$$X_i \overset{d}{=} a_i + b_i W_i, \qquad \text{for } i = 1, 2,$$

where the W_i's are symmetric and identically distributed random variables, which implies that the function g is always monotone, and the sign of its derivative depends upon the sign of $b_1 - b_2$. Chaoubi et al. (2020) derive decompositions of the VaR and TVaR of counter-monotonic sums in the case of non-decreasing functions g, as well as in the more challenging case of concave/convex functions g.

3.2   The set E_x

In order to derive decomposition formulas for risk measures of counter-monotonic sums that hold for general marginal distributions, a closer analysis has to be performed on the behavior of the function g. A key component in the derivations is the set E_x introduced hereafter. Note that throughout the paper, the set E_x is associated with the function g, even though it is not explicitly included in the notation.

Definition 3.2 *For any $x \in \left(x^{\min}, x^{\max}\right)$, with $x^{\max} = \sup_{u \in [0,1]} g(u)$ and $x^{\min} = \inf_{u \in [0,1]} g(u)$, and $j = 1, ..., N_x$, the element $u_{x,j} \in (0,1)$ belongs to E_x if and only if there exist $a_j, b_j, c_j \in [0,1]$ with $a_j < b_j < c_j$ such that one of the following conditions is satisfied:*

1. $g(u) < x$ *for* $u \in (a_j, b_j)$, *and* $g(u) = x$ *for* $u \in [b_j, u_{x,j})$, *and* $g(u) > x$ *for* $u \in (u_{x,j}, c_j)$.

2. $g(u) > x$ *for* $u \in (a_j, b_j)$, *and* $g(u) = x$ *for* $u \in [b_j, u_{x,j})$, *and* $g(u) < x$ *for* $u \in (u_{x,j}, c_j)$.

Definition 3.2 states that for $u_{x,j}$ to be an element of E_x, the function g must cross the level x, where the crossing occurs from below under Condition 1 and from above under Condition 2. If the function g reaches x without crossing it, no elements would be counted in E_x. In case the function g equals x before crossing it, then the element that belongs to E_x is the last one before the crossing occurs.

Figure 1 provides a general illustration of how the set E_x is defined. For a given $x \in (x^{\min}, x^{\max})$, the set E_x contains the elements $u_{x,1}, u_{x,2}, u_{x,3}, u_{x,4}$ and $u_{x,5}$. For $u_{x,1}, u_{x,2}$ and $u_{x,3}$, the function is locally continuous and strictly monotone. In particular, Condition 1 of Definition 3.2 is satisfied for $u_{x,1}$ and $u_{x,3}$, where the level x is crossed from below, and Condition 2 is satisfied for $u_{x,2}$, where x is crossed from above. The function g is left-continuous and has a jump in $u_{x,4}$. Inequality $g(u_{x,4}+) < x < g(u_{x,4})$ holds, and Condition 2 is satisfied, where $g(u_{x,4}+)$ is the right-limit of g in $u_{x,4}$. Note that $g(u_{x,4}) \neq x$. The points q_1 and q_2 in Figure 1 do not belong to the set E_x. Although $g(q_1) = g(q_2) = x$, the function g does not cross x immediately after these points. The function g is flat on the interval $[q_2, u_{x,5}]$, with $g(u) = x$ for all $u \in [q_2, u_{x,5}]$. However, only the point $u_{x,5}$ from that interval belongs to the set E_x, since for this point Condition 1 of Definition 3.2 is satisfied.
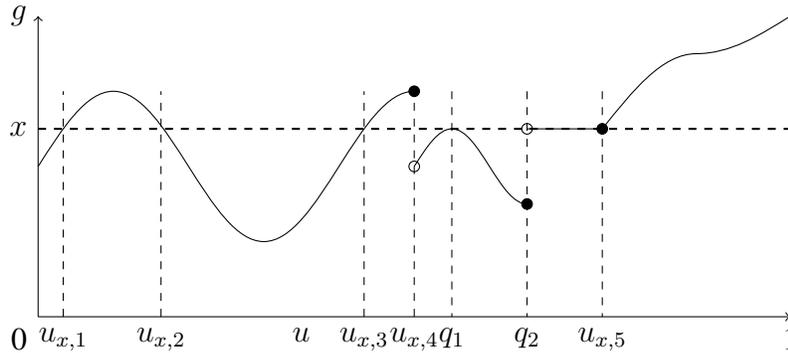

Figure 1: Illustration of the function g and the set E_x, where $E_x = \{u_{x,1}, u_{x,2}, u_{x,3}, u_{x,4}, u_{x,5}\}$, and q_1 and q_2 do not belong to E_x.

The set E_x is not empty for $x \in (x^{\min}, x^{\max})$, and is at most countably infinite (i.e. N_x can be infinity), as long as S^l is not degenerate. The trivial case where S^l is degenerate is not discussed here. If the function g is monotone, the set E_x contains only a single element for any $x \in (x^{\min}, x^{\max})$. In this case, Condition 1 corresponds to non-decreasingness whereas Condition 2 corresponds to non-increasingness. If g is either convex or concave, the set E_x contains at most two elements, where these two elements are equidistant from the minimum of

g when g is symmetric; closed-form expressions of the elements of E_x in these cases can be found in Chaoubi et al. (2020).

A typical situation where the set E_x contains many elements, and that motivates the present study, is when one of the two random variables is continuous (e.g. investment fund value) and the other is discrete (e.g. number of claims). Suppose for instance that X_1 is continuous and X_2 is discrete. The inverse $F_{X_1}^{-1}$ is a strictly increasing function, whereas the inverse $F_{X_2}^{-1}$ is non-decreasing. As a consequence, the function g increases on the flat segments of $F_{X_2}^{-1}$, and has downward jumps due to the jumps of $F_{X_2}^{-1}$. The situation where the set E_x contains many elements can also arise when X_1 and X_2 are both continuous, but exhibit some features such as skewness and excess kurtosis.

Figure 2 illustrates the function g with two examples. The top panel of Figure 2 displays the function g when X_1 and X_2 follow a Gamma distribution with shape parameters 4 and 3 respectively, and scale parameter 1 in both cases. The combination of these two distributions results in a concave and asymmetric function g. The horizontal line x corresponds to the quantile of the counter-monotonic sum S^l at the level 0.95, which is equal to 8.94. Due to the concave shape of g, the set $E_{8.94}$ contains two elements. The asymmetry of g implies that the elements of $E_{8.94}$ are not equidistant from the point where the minimum of g is attained.

The bottom panel of Figure 2 displays the case where X_1 is assumed to follow a Gamma distribution with shape parameter 5 and scale parameter 1, and X_2 is assumed to follow a Poisson distribution with rate parameter 5. The shape of the function g is rather complex. Moreover, for $x = 9.85$, which is the median of the counter-monotonic sum, the set $E_{9.85}$ has 12 elements.

The remainder of the paper contains the main results. Namely, the decomposition of the VaR, the TVaR, and the upper tail transform of the counter-monotonic sum are derived. The two examples of the function g from Figure 2 are used to illustrate the results. In the sequel, the notation $u_{x,j}$ and N_x is used to denote the elements of the set E_x when this set is defined for any x. When x stands for the inverse $F_{S^l}^{-1}(p)$, the notation $u_{p,j}$ is used instead of $u_{F_{S^l}^{-1}(p),j}$ for the elements of $E_{F_{S^l}^{-1}(p)}$, and the notation N_p is used instead of $N_{F_{S^l}^{-1}(p),j}$. Similarly, the notations $u_{p,j}^{\alpha}$ and N_p^{α} are used instead of $u_{F_{S^l}^{-1(\alpha)}(p),j}$ and $N_{F_{S^l}^{-1(\alpha)}(p),j}$ when x stands for the generalized inverse $F_{S^l}^{-1(\alpha)}(p)$.
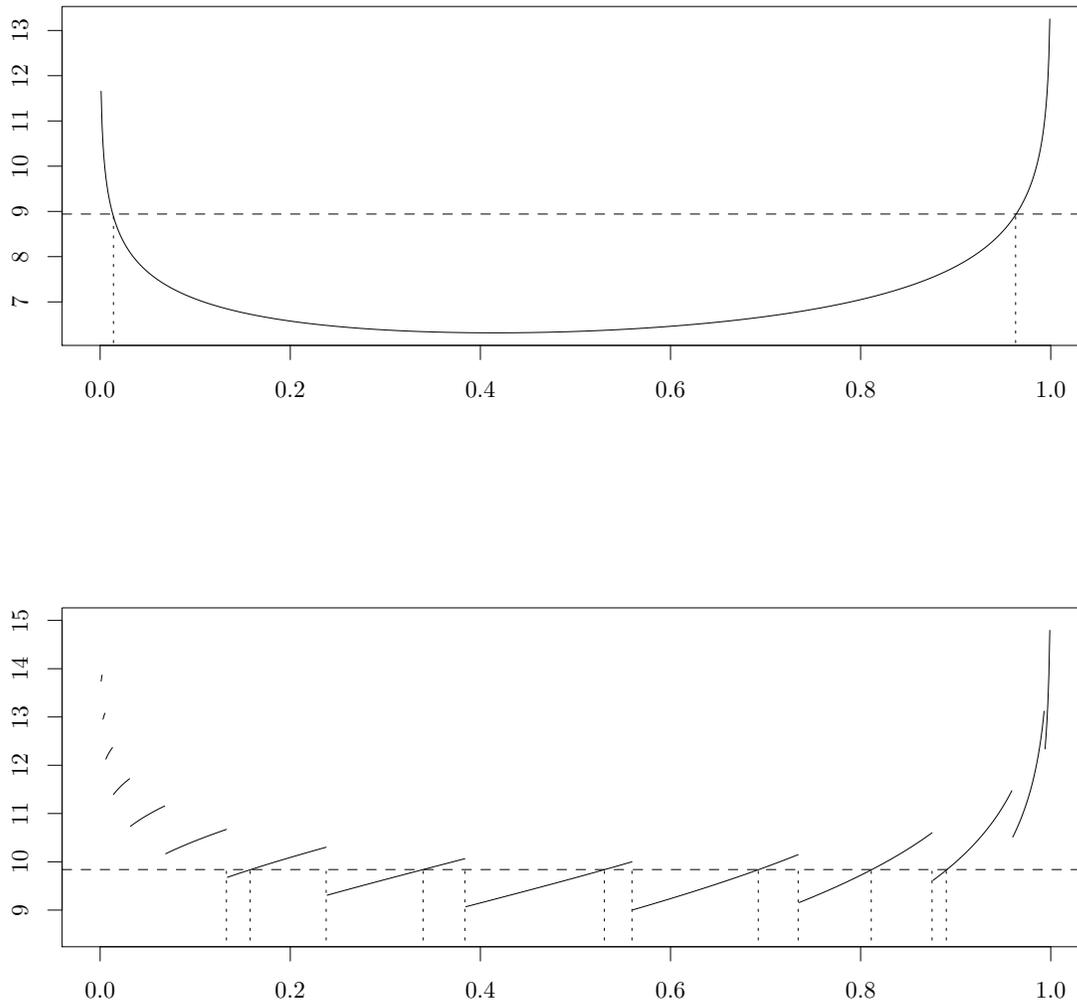
Figure 2: Two examples of the function g. In the top panel, both X_1 and X_2 follow a Gamma distribution with shape parameters 4 and 3, respectively, and scale parameter equal to 1 for both. In the bottom panel, X_1 follows a Gamma distribution with shape parameter 5 and scale parameter 1, and X_2 follows a Poisson distribution with rate parameter 5.

4 Value-at-Risk of the counter-monotonic sum

4.1 Main results

The decomposition of the VaR is closely related to the definition of the set E_x. For $p \in (0, 1)$, let $u_{p,1} < u_{p,2} < ... < u_{p,N_p}$ be the N_p elements of $E_{F_{S^l}^{-1}(p)}$. Recall that by definition, for any

$j \in \{1, ..., N_p\}$, the function g crosses the level $F_{S^l}^{-1}(p)$ at $u_{p,j}$. If g has a jump in $u_{p,j}$, then $g(u_{p,j}) \neq F_{S^l}^{-1}(p)$. If g is continuous in $u_{p,j}$, then $g(u_{p,j}) = F_{S^l}^{-1}(p)$. Therefore, the following equality holds:

$$\mathrm{VaR}_p[S^l] = \mathrm{VaR}_{u_{p,j}}[X_1] + \mathrm{VaR}_{1-u_{p,j}}[X_2], \tag{4.1}$$

for any $p \in (0,1)$, where $u_{p,j}$ is one of the N_p elements of $E_{F_{S^l}^{-1}(p)}$ in which g does not jump.

If g is continuous in more than one element of the set $E_{F_{S^l}^{-1}(p)}$, then the VaR of S^l admits multiple decompositions, i.e. (4.1) holds for all elements $u_{p,j}$ in which g does not jump. In fact, the decomposition holds for all $u \in (0,1)$ such that $g(u) = F_{S^l}^{-1}(p)$, and is not restricted to the elements of the set $E_{F_{S^l}^{-1}(p)}$. The decomposition of VaR when g is not monotone is therefore not necessarily unique.

If g is not continuous in a given element $u_{p,j}$ of $E_{F_{S^l}^{-1}(u_{p,j})}$, then the decomposition (4.1) does not hold for that element, i.e. $g(u_{p,j}) \neq F_{S^l}^{-1}(p)$. Nevertheless, it is still possible to find an α-quantile of X_1 and a corresponding $(1-\alpha)$-quantile of X_2 such that $\mathrm{VaR}_p\left[S^l\right]$ is equal to their sum. This result is stated in the following lemma; see Appendix A.1 for a proof. Note that the decomposition of the VaR of S^l is obtained by setting $x = F_{S^l}^{-1}(p)$.

Lemma 4.1 *For any $x \in \left(x^{\min}, x^{\max}\right)$ and any element $u_{x,j}$ in E_x there exists an $\alpha_{x,j}$ satisfying:*

$$x = F_{X_1}^{-1(\alpha_{x,j})}(u_{x,j}) + F_{X_2}^{-1(1-\alpha_{x,j})}(1 - u_{x,j}). \tag{4.2}$$

Moreover:

$$\alpha_{x,j} = \begin{cases} \frac{x - g(u_{x,j}-)}{g(u_{x,j}+) - g(u_{x,j}-)}, & \text{if } g(u_{x,j}+) \neq g(u_{x,j}-), \\ 0, & \text{if } g(u_{x,j}+) = g(u_{x,j}-). \end{cases} \tag{4.3}$$

where $g(u_{x,j}+)$ and $g(u_{x,j}-)$ are the right and left limits, respectively, of the function g in $u_{x,j}$.

If g is continuous in $u_{x,j}$, i.e. $g(u_{x,j}-) = g(u_{x,j}+) = g(u_{x,j})$, Equality (4.2) holds for any choice of $\alpha_{x,j}$, and more importantly, the equality $g(u_{x,j}) = x$ holds. If g has a jump in $u_{x,j}$, i.e. $g(u_{x,j}-) \neq g(u_{x,j}+)$, the choice of $\alpha_{x,j}$ is unique. The value of $\alpha_{x,j}$ locates x within the discontinuity, and quantifies the distance between x and the function g.

4.2 Illustration

Consider the first example where X_1 and X_2 both follow a Gamma distribution, i.e. the top panel on Figure 2. For $x = F_{S^l}^{-1}(0.95)$, the set $E_{F_{S^l}^{-1}(0.95)}$ contains the following two elements:

$$u_{p,1} = 0.01328 \quad \text{and} \quad u_{p,2} = 0.96358. \tag{4.4}$$

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Since the function g is continuous, the decomposition in (4.1) holds for both $u_{p,1}$ and $u_{p,2}$.

Consider now the second example where X_1 follows a Gamma distribution and X_2 follows a Poisson distribution, i.e. the bottom panel on Figure 2. For $x = F_{S^l}^{-1}(0.5)$, the set $E_{F_{S^l}^{-1}(0.5)}$ contains the following twelve elements:

$$
\begin{aligned}
u_{p,1} &= 0.13337, & u_{p,2} &= 0.15843, & u_{p,3} &= 0.23781, \\
u_{p,4} &= 0.33971, & u_{p,5} &= 0.38404, & u_{p,6} &= 0.53079, \\
u_{p,7} &= 0.55951, & u_{p,8} &= 0.69279, & u_{p,9} &= 0.73497, \\
u_{p,10} &= 0.81179, & u_{p,11} &= 0.87535, & u_{p,12} &= 0.89076.
\end{aligned}
\tag{4.5}
$$

Figure 2 also shows that the function g has jumps in $u_{p,1}$, $u_{p,3}$, $u_{p,5}$, $u_{p,7}$, $u_{p,9}$ and $u_{p,11}$. For these elements, the decomposition (4.1) does not hold. Instead, the result of Lemma 4.1 has to be used. In particular, from (4.3), it follows that the $\alpha_{p,j}$'s required for (4.2) are given by $\alpha_{p,1} = 0.83598$, $\alpha_{p,3} = 0.46330$, $\alpha_{p,5} = 0.22700$, $\alpha_{p,7} = 0.16108$, $\alpha_{p,9} = 0.31461$ and $\alpha_{p,11} = 0.76530$. For the remaining elements, the decomposition (4.1) holds.

5 Tail Value-at-Risk of the counter-monotonic sum

5.1 Main results

The following theorem provides a decomposition for the TVaR of S^l in terms of marginal TVaR's and LTVaR's; see Appendix A.2 for a proof.

Theorem 5.1 *For any $p \in (0,1)$ and $\alpha \in [0,1]$, let $u_{p,1}^\alpha < ... < u_{p,N_p^\alpha}^\alpha$ be the N_p^α ordered elements of the set $E_{F_{S^l}^{-1(\alpha)}(p)}$. The* Tail Value-at-Risk *at the level p of S^l can be expressed as follows:*

$$
\text{TVaR}_p\left[S^l\right] = \begin{cases} \frac{1}{1-p}t_1^\alpha(p), & \text{if } g(u) \leq F_{S^l}^{-1(\alpha)}(p) \text{ for } u \in \left(0, u_{p,1}^\alpha\right), \\ \frac{1}{1-p}t_2^\alpha(p), & \text{if } g(u) \geq F_{S^l}^{-1(\alpha)}(p) \text{ for } u \in \left(0, u_{p,1}^\alpha\right), \end{cases}
\tag{5.1}
$$

where

$$
\begin{aligned}
t_1^\alpha(p) &= (1 - u_{p,1}^\alpha)\left(\text{TVaR}_{u_{p,1}^\alpha}[X_1] + \text{LTVaR}_{1-u_{p,1}^\alpha}[X_2]\right) - \mathcal{T}_{p,N_p^\alpha}^\alpha + F_{S^l}^{-1(\alpha)}(p)\left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right), \\
t_2^\alpha(p) &= u_{p,1}^\alpha\left(\text{LTVaR}_{u_{p,1}^\alpha}[X_1] + \text{TVaR}_{1-u_{p,1}^\alpha}[X_2]\right) + \mathcal{T}_{p,N_p^\alpha}^\alpha + F_{S^l}^{-1(\alpha)}(p)\left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right),
\end{aligned}
\tag{5.2}
$$

and

$$\mathcal{T}^{\alpha}_{p,N^{\alpha}_p} = \sum_{j=2}^{N^{\alpha}_p}(-1)^j(1 - u^{\alpha}_{p,j})\left(\text{TVaR}_{u^{\alpha}_{p,j}}[X_1] + \text{LTVaR}_{1-u^{\alpha}_{p,j}}[X_2]\right),$$

$$\mathcal{D}^{\alpha}_{p,N^{\alpha}_p} = \sum_{j=2}^{N^{\alpha}_p}(-1)^j\left(1 - u^{\alpha}_{p,j}\right),$$

(5.3)

and $\sum_{j=2}^{1} = 0$ by convention.

For a given $p \in (0,1)$, each choice of $\alpha \in [0,1]$ will result in a specific decomposition of the Tail Value-at-Risk. In case the marginal cdf's F_{X_1} and F_{X_2} are continuous, all the generalized inverses in (5.2) coincide and the decomposition of the TVaR is unique. The choice $\alpha = 0$ leads to a decomposition in terms of the usual inverse $F^{-1}_{S^l}(p)$.

The decomposition of the TVaR of the sum in the counter-monotonic case involves more terms than the decomposition formula of the TVaR in the comonotonic case. In particular, on top of the linear combination involving the first element $u^{\alpha}_{p,1}$ of the set $E_{F^{-1(\alpha)}_{S^l}(p)}$, the sum $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$ and the terms $F^{-1(\alpha)}_{S^l}(p)\left(u^{\alpha}_{p,1} + \mathcal{D}^{\alpha}_{p,N^{\alpha}_p} - p\right)$ and $F^{-1(\alpha)}_{S^l}(p)\left(1 - u^{\alpha}_{p,1} - \mathcal{D}^{\alpha}_{p,N^{\alpha}_p} - p\right)$ are also required.

The number of components of the sum $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$ depends on the number of elements in the set $E_{F^{-1(\alpha)}_{S^l}(p)}$, and $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$ vanishes when the function g is monotone. The more complex the shape of g is, the more terms may appear in $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$. A noteworthy remark is that the notation $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$ is only used to synthesize the result. This notation does not imply that the contribution of this sum to the TVaR of S^l is less important than the contribution of the first term containing the marginal TVaR and LTVaR related to the first element $u^{\alpha}_{p,1}$. In fact, as it is shown in the numerical examples below, the sum $\mathcal{T}^{\alpha}_{p,N^{\alpha}_p}$ can capture a substantial part of $\text{TVaR}_p[S^l]$.

The terms $F^{-1(\alpha)}_{S^l}(p)\left(u^{\alpha}_{p,1} + \mathcal{D}^{\alpha}_{p,N^{\alpha}_p} - p\right)$ and $F^{-1(\alpha)}_{S^l}(p)\left(1 - u^{\alpha}_{p,1} - \mathcal{D}^{\alpha}_{p,N^{\alpha}_p} - p\right)$ in (5.2) arise because the quantile function $F^{-1(\alpha)}_{S^l}$ may be constant in the neighborhood of p, i.e. $F^{-1(\alpha)}_{S^l}$ is 'flat' in p and hence, $F_{S^l}\left(F^{-1(\alpha)}_{S^l}(p)\right)$ is not necessarily equal to p. For the values of p such that $F^{-1(\alpha)}_{S^l}$ is strictly increasing in p, it can be shown that these terms equal 0.

The following lemma proves this result, thereby providing a link between the elements of the set $E_{F^{-1(\alpha)}_{S^l}(p)}$ and p in case the quantile function $F^{-1(\alpha)}_{S^l}$ is strictly increasing in p; see Appendix A.3 for a proof.

Lemma 5.1 *For any $p \in (0,1)$ and $\alpha \in [0,1]$, suppose that the function $F^{-1(\alpha)}_{S^l}$ is strictly increasing in p, and let $u^{\alpha}_{p,1} < ... < u^{\alpha}_{p,N^{\alpha}_p}$ be the N^{α}_p ordered elements of the set $E_{F^{-1(\alpha)}_{S^l}(p)}$.*

Then:

$$p = \begin{cases} u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha, & \text{if} \quad g(u) \leq F_{S^l}^{-1(\alpha)}(p) \quad \text{for} \quad u \in (0, u_{p,1}^\alpha), \\ 1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha, & \text{if} \quad g(u) \geq F_{S^l}^{-1(\alpha)}(p) \quad \text{for} \quad u \in (0, u_{p,1}^\alpha), \end{cases} \tag{5.4}$$

where $\mathcal{D}_{p,N_p^\alpha}^\alpha$ is given in (5.3).

Lemma 5.1 shows that the situation where $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p has two consequences. The first consequence is that the expression of $\text{TVaR}_p\left[S^l\right]$ can be simplified because $F_{S^l}^{-1(\alpha)}(p)\left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right) = 0$ and $F_{S^l}^{-1(\alpha)}(p)\left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right) = 0$. The second consequence is that, for $N_p^\alpha \geq 2$, it is sufficient to determine $N_p^\alpha - 1$ elements, whereas for $N_p^\alpha = 1$, $u_{p,1}^\alpha$ is either equal to p or to $1 - p$.

The decomposition of TVaR in Theorem 5.1 will be further simplified in the next section, where a link between $t_1^\alpha(p)$ and $t_2^\alpha(p)$ is established using the decomposition of the upper tail transform.

5.2 The case where $N_p^\alpha = 1$ and $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p

Suppose that $N_p^\alpha = 1$, which includes the case where g is monotone. Additionally, suppose that $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p. This situation leads to simple decompositions of $\text{TVaR}_p\left[S^l\right]$, which are provided in the following corollary. The result follows directly from Theorem 5.1 and Lemma 5.1. In particular, since the generalized quantile function $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p, then from Lemma 5.1:

$$u_{p,1}^\alpha = \begin{cases} p, & \text{if} \quad g(0) \leq g(1), \\ 1 - p, & \text{if} \quad g(0) \geq g(1). \end{cases}$$

Note that for $N_p^\alpha = 1$, the case $g(u) \leq F_{S^l}^{-1(\alpha)}(p)$ for $u \in (0, u_{p,1}^\alpha)$ is equivalent with $g(0) \leq g(1)$, whereas the case $g(u) \geq F_{S^l}^{-1(\alpha)}(p)$ for $u \in (0, u_{p,1}^\alpha)$ is equivalent with $g(0) \geq g(1)$.

Corollary 1 *For any $p \in (0,1)$ and $\alpha \in [0,1]$, if $N_p^\alpha = 1$ and the function $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p, then:*

$$\text{TVaR}_p\left[S^l\right] = \begin{cases} \text{TVaR}_p[X_1] + \text{LTVaR}_{1-p}[X_2], & \text{if} \quad g(0) \leq g(1), \\ \text{LTVaR}_{1-p}[X_1] + \text{TVaR}_p[X_2], & \text{if} \quad g(0) \geq g(1). \end{cases} \tag{5.5}$$

The case where $g(0) \leq g(1)$ leads to a decomposition which is consistent with that of Chaoubi et al. (2020) for strictly increasing functions g. Thus, Corollary 1 generalizes their expression

by stating that their decomposition does not require the function g to be globally monotone, nor both marginal random variables to be continuous. More specifically, only $N_p^\alpha = 1$ and local strict increasingness of the quantile $F_{S^l}^{-1(\alpha)}$ in p are sufficient to derive decomposition (5.5).

Interestingly, under these conditions, the dependence uncertainty spread (i.e. the difference between the upper and lower bounds) can be expressed in terms of the TVaR and LTVaR of only one of the random variables. In particular, using the decomposition of the upper bound $\text{TVaR}_p[S^u]$, it follows that if $N_p^\alpha = 1$ and $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p, then:

$$\text{TVaR}_p[S^u] - \text{TVaR}_p[S^l] = \begin{cases} \text{TVaR}_p\left[X_2\right] - \text{LTVaR}_{1-p}\left[X_2\right], & \text{if } g(0) \le g(1), \\ \text{TVaR}_p\left[X_1\right] - \text{LTVaR}_{1-p}\left[X_1\right], & \text{if } g(0) \ge g(1). \end{cases} \tag{5.6}$$

Thus, the dependence uncertainty spread of $\text{TVaR}_p\left[S^l\right]$ depends on the tail distribution of one of the two random variables only.

Suppose that the function g is strictly monotone, which is a special case of the conditions $N_p^\alpha = 1$ and $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p. Since both the TVaR and the LTVaR are non-decreasing functions of p, it follows from (5.6) that the dependence uncertainty spread is a non-decreasing function of the level p. This result implies that in case g is strictly monotone, a more conservative risk measurement (i.e. higher levels of p) leads to a larger dependence uncertainty gap. This is in general not the case, as the dependence uncertainty spread can be a non-monotone function of p. This can be seen by combining the decomposition of the upper bound of $\text{TVaR}_p[S]$ in (2.11) and the decomposition of the corresponding lower bound given in Theorem 5.1.

5.3   Illustration

The decomposition of the TVaR is illustrated using the examples from Figure 2, with a particular emphasis on the components of the sum $\mathcal{T}_{p,N_p^\alpha}^\alpha$. The parameter α is set to 0, i.e. $F_{S^l}^{-1(\alpha)}(p) = F_{S^l}^{-1}(p)$, and for simplicity, the superscript α is dropped from all relevant quantities.

In both cases of Figure 2, it holds that $g(u) \ge F_{S^l}^{-1}(p)$ for $u \in (0, u_{p,1})$. This means that the decomposition is given by $t_2(p)$. Further, as noted previously, Lemma 5.1 leads to $p = 1 - u_{p,1} - \mathcal{D}_{p,N_p}$, which is due to the fact that g does not have flat parts, and hence, the last term in $t_2(p)$ vanishes. This can also be verified numerically using the values given in (4.4) and (4.5). Therefore, in both cases, the expression of $\text{TVaR}_p\left[S^l\right]$ is given by:

$$\text{TVaR}_p\left[S^l\right] = u_{p,1}\left(\text{LTVaR}_{u_{p,1}}\left[X_1\right] + \text{TVaR}_{1-u_{p,1}}\left[X_2\right]\right) + \mathcal{T}_{p,N_p},$$

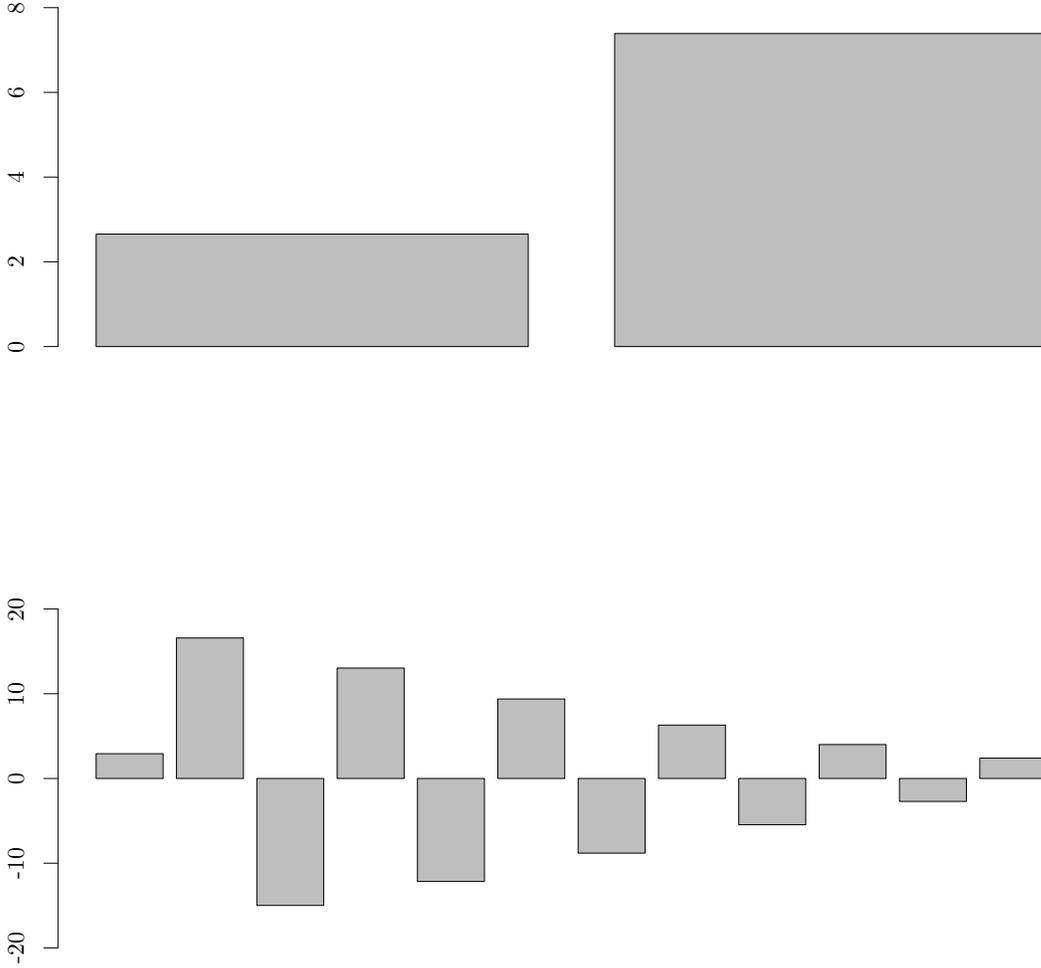where $\mathcal{T}_{p,N_p}$ is given in (5.3) with $\alpha = 0$.

Figure 3: Components of the TVaR decomposition for the two examples of the function g from Figure 2 in the order they appear in the decomposition.

The set $E_{F_{Sl}^{-1}(0.95)}$ contains two elements in the first example with two Gamma distributed random variables. This means that $N_p = 2$, and that $t_2(0.95)$ has two components. The first component is $\frac{u_{p,1}}{1-p} \left(\text{LTVaR}_{u_{p,1}}[X_1] + \text{TVaR}_{1-u_{p,1}}[X_2] \right)$ and the second component is $\mathcal{T}_{p,N_p} = \frac{1-u_{p,2}}{1-p} \left(\text{TVaR}_{u_{p,2}}[X_1] + \text{LTVaR}_{1-u_{p,2}}[X_2] \right)$. Both are displayed on the top panel of Figure 3, where the first component is on the left and the second component is on the right. The figure shows that $\mathcal{T}_{0.95,N_p}$ is substantially larger than the first component, and largely contributes to the

TVaR of the counter-monotonic difference.

In the second example involving Gamma and Poisson random variables, the set $E_{F_{S^l}^{-1}(0.5)}$ contains twelve elements, which means that $t_2(0.5)$ has twelve components. The first component is $\frac{u_{p,1}}{1-p} \left(\text{LTVaR}_{u_{p,1}}[X_1] + \text{TVaR}_{1-u_{p,1}}[X_2] \right)$. The remaining eleven components with alternating sign are $(-1)^j \frac{1-u_{p,j}}{1-p} \left(\text{TVaR}_{u_{p,j}}[X_1] + \text{LTVaR}_{1-u_{p,j}}[X_2] \right)$, for $j = 2, ..., 12$, where their sum is $\mathcal{T}_{p,N_p}$. These twelve terms are displayed in the bottom panel of Figure 3 in the order in which they appear in the expression, from left to right. Note that the decomposition leads to a TVaR of the counter-monotonic sum equal to 10.505. In this example, the first component does not contribute as much as the remaining ones. Further, from the second onward, the amplitudes are decreasing. This is due to the fact that the components are weighted by $(1 - u_{p,j})$, and hence, additional terms weighted by values of $u_{p,j}$ close to 1 have less importance.

6 Upper tail transform of the counter-monotonic sum

6.1 Main results

The following theorem provides a decomposition formula for the upper tail transform of the counter-monotonic stop-loss premium; see Appendix A.4 for a proof.

Theorem 6.1 *For any $x \in \left(x^{\min}, x^{\max} \right)$, let $u_{x,1} < ... < u_{x,N_x}$ be the N_x ordered elements of the set E_x. The* upper tail transform *at the level x of S^l can be expressed as follows:*

$$
\pi_{S^l}(x) = \begin{cases} s_1(x), & \text{if } g(u) \leq x \ \text{for} \ u \in (0, u_{x,1}), \\ s_2(x), & \text{if } g(u) \geq x \ \text{for} \ u \in (0, u_{x,1}), \end{cases}
\tag{6.1}
$$

where:

$$
\begin{aligned}
s_1(x) &= \pi_{X_1} \left(F_{X_1}^{-1}(u_{x,1}) \right) - \lambda_{X_2} \left(F_{X_2}^{-1}(1 - u_{x,1}) \right) - \mathcal{S}_{x,N_x} + (1 - u_{x,1})(g(u_{x,1}) - x) - \mathcal{J}_{x,N_x}, \\
s_2(x) &= \pi_{X_2} \left(F_{X_2}^{-1}(1 - u_{x,1}) \right) - \lambda_{X_1} \left(F_{X_1}^{-1}(u_{x,1}) \right) + \mathcal{S}_{x,N_x} + u_{x,1}(g(u_{x,1}) - x) + \mathcal{J}_{x,N_x},
\end{aligned}
\tag{6.2}
$$

with

$$
\begin{aligned}
\mathcal{S}_{x,N_x} &= \sum_{j=2}^{N_x} (-1)^j \left(\pi_{X_1} \left(F_{X_1}^{-1}(u_{x,j}) \right) - \lambda_{X_2} \left(F_{X_2}^{-1}(1 - u_{x,j}) \right) \right), \\
\mathcal{J}_{x,N_x} &= \sum_{j=2}^{N_x} (-1)^j (1 - u_{x,j})(g(u_{x,j}) - x),
\end{aligned}
\tag{6.3}
$$

and $\sum_{j=2}^{1} = 0$ by convention.

It follows that the stop-loss premium of the counter-monotonic sum is equal to the difference of an upper tail transform of one component and a lower tail transform of the other component, but analogously to the decomposition of $\text{TVaR}_p\left[S^l\right]$, additional terms appear that depend on the shape of the function g. In particular, the sum $\mathcal{S}_{x,N_x}$ accounts for the fact that the function g is not necessarily monotone, and vanishes when $N_x = 1$, i.e. $\mathcal{S}_{x,1} = 0$. The sum $\mathcal{J}_{x,N}$ as well as $(1 - u_{x,1})\left(g(u_{x,1}) - x\right)$ and $u_{x,1}\left(g(u_{x,1}) - x\right)$ appear in case the function g has a jump in one of the elements of E_x. In general, these terms vanish only when g is continuous in all the elements of E_x. This is because if g is continuous in an element $u_{x,j}$, then $g(u_{x,j}) = x$.

The following theorem provides an alternative expression for $s_1(x)$ and $s_2(x)$. This alternative decomposition expresses the retentions in terms of the generalized inverses of X_1 and X_2 instead of left inverses $F_{X_1}^{-1}$ and $F_{X_2}^{-1}$. A proof of this result is given in Appendix A.5.

Theorem 6.2 *For any* $x \in \left(x^{\min}, x^{\max}\right)$, *let* $u_{x,1} < ... < u_{x,N_x}$ *be the* N_x *ordered elements of the set* E_x. *The* upper tail transform *at the level* x *of* S^l *can be expressed as follows:*

$$\pi_{S^l}(x) = \begin{cases} s_1(x), & \text{if } g(u) \leq x \ \text{ for } \ u \in (0, u_{x,1}), \\ s_2(x), & \text{if } g(u) \geq x \ \text{ for } \ u \in (0, u_{x,1}), \end{cases} \tag{6.4}$$

where:

$$\begin{aligned} s_1(x) &= \pi_{X_1}\left(F_{X_1}^{-1(\alpha_{x,1})}(u_{x,1})\right) - \lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,1})}(1 - u_{x,1})\right) - \mathcal{S}_{x,N_x}, \\ s_2(x) &= \pi_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,1})}(1 - u_{x,1})\right) - \lambda_{X_1}\left(F_{X_1}^{-1(\alpha_{x,1})}(u_{x,1})\right) + \mathcal{S}_{x,N_x}, \end{aligned} \tag{6.5}$$

with

$$\mathcal{S}_{x,N_x} = \sum_{j=2}^{N_x} (-1)^j \left(\pi_{X_1}\left(F_{X_1}^{-1(\alpha_{x,j})}(u_{x,j})\right) - \lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,j})}(1 - u_{x,j})\right)\right), \tag{6.6}$$

and $\sum_{j=2}^{1} = 0$ *by convention, and* $\alpha_{x,j}$'s *are determined from:*

$$x = F_{X_1}^{-1(\alpha_{x,j})}(u_{x,j}) + F_{X_2}^{-1(1-\alpha_{x,j})}(1 - u_{x,j}), \tag{6.7}$$

for $j = 1, 2, \ldots, N_x$.

In contrast with the expression in Theorem 6.1, the alternative expression in Theorem 6.2 does not have the terms $\mathcal{J}_{x,N_x}$ nor $(1 - u_{x,1})\left(g(u_{x,1}) - x\right)$ and $u_{x,1}\left(g(u_{x,1}) - x\right)$. In counterpart, it requires determining the generalized inverses of X_1 and X_2.

It can be proven that $s_1(x) + s_2(x) = \mathbb{E}\left[S^l\right] - x$. Combining this equality with the parity (2.6), the decomposition of the lower bound $\pi_{S^l}(x)$ can be further simplified. In particular, if $s_1(x)$

is the decomposition of the upper tail transform $\pi_{S^l}(x)$, then $-s_2(x)$ is the decomposition of the corresponding lower tail transform $\lambda_{S^l}(x)$, which means that only one of the two quantities is positive. Hence, the decomposition of $\pi_{S^l}(x)$ is always the highest of $s_1(x)$ and $s_2(x)$. This simplification applies to the TVaR decomposition as well. The following lemma formalizes the result; see Appendix A.6 for more details on the proof.

Lemma 6.1 *For any $p \in (0,1)$, any $\alpha \in [0,1]$ and any $x \in (x^{\min}, x^{\max})$, the* Tail Value-at-Risk *at the level p and the upper tail transform at the level x of S^l can be expressed as follows:*

$$\text{TVaR}_p\left[S^l\right] = \frac{1}{1-p} \max\left\{t_1^\alpha(p), t_2^\alpha(p)\right\},$$

$$\pi_{S^l}(x) = \max\{s_1(x), s_2(x)\},$$

where $t_1^\alpha(p)$ and $t_2^\alpha(p)$ are given in (5.2), and $s_1(x)$ and $s_2(x)$ are given in (6.2).

6.2 The case where $N_x = 1$

For $x \in (x^{\min}, x^{\max})$, suppose that the marginal cdf's F_{X_1} and F_{X_2} are such that $N_x = 1$, i.e. the set E_x contains a single element $u_{x,1}$. The following corollary shows that the decomposition of the stop-loss premium in case $N_x = 1$ can be expressed in terms of $F_{S^l}(x)$. Thus, the case $N_x = 1$ leads to decompositions which are similar to the decomposition of the upper tail transform of the comonotonic sum in (2.12). Appendix A.7 contains the proof. Recall that for $N_x = 1$, the case $g(u) \leq x$ for $u \in (0, u_{x,1})$ is equivalent with $g(0) \leq g(1)$, whereas the case $g(u) \geq x$ for $u \in (0, u_{x,1})$ is equivalent with $g(0) \geq g(1)$.

Corollary 2 *For any $x \in (x^{\min}, x^{\max})$, if $N_x = 1$, the* upper tail transform *at the level x of S^l can be expressed as follows:*

$$\pi_{S^l}(x) = \begin{cases} \pi_{X_1}\left(F_{X_1}^{-1(\alpha_x)}\left(F_{S^l}(x)\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_x)}\left(1 - F_{S^l}(x)\right)\right), & \text{if } g(0) \leq g(1), \\ \pi_{X_2}\left(F_{X_2}^{-1(1-\alpha_x)}\left(F_{S^l}(x)\right)\right) - \lambda_{X_1}\left(F_{X_1}^{-1(\alpha_x)}\left(1 - F_{S^l}(x)\right)\right), & \text{if } g(0) \geq g(1), \end{cases}$$

where α_x is determined from:

$$x = \begin{cases} F_{X_1}^{-1(\alpha_x)}\left(F_{S^l}(x)\right) + F_{X_2}^{-1(1-\alpha_x)}\left(1 - F_{S^l}(x)\right), & \text{if } g(0) \leq g(1), \\ F_{X_2}^{-1(1-\alpha_x)}\left(F_{S^l}(x)\right) + F_{X_1}^{-1(\alpha_x)}\left(1 - F_{S^l}(x)\right), & \text{if } g(0) \geq g(1). \end{cases}$$

Note that unlike for the TVaR, the dependence uncertainty spread of $\pi_{S^l}(x)$ in case $N_x = 1$ is not determined by one of the two random variables as in (5.6).

6.3 Illustration

The importance of each term in $\mathcal{S}_{x,N_x}$ is similar to that of the terms in $\mathcal{T}_{p,N_p^\alpha}^\alpha$, which means that the discussion from the illustration of Section 5 remains valid. This numerical illustration focuses on the importance of the terms due to the jumps of the function g. Only the second example where X_2 follows a Poisson distribution is discussed, because there are no jumps in the other example. In order to illustrate the term due to the jumps, the decomposition from Theorem 6.1 is used. For $x = F_{S^l}^{-1}(0.5)$, the bottom panel of Figure 2 shows that $g(u) \geq x$ for $u \in (0, u_{x,1})$. Thus, the decomposition of the stop-loss premium is given by $s_2(x)$, and the component under interest is $u_{x,1}(g(u_{x,1}) - x) + \mathcal{J}_{x,N_x}$.

Using the values of the elements of E_x, it follows from Theorem 6.1 that the stop-loss premium at the level $x = F_{S^l}^{-1}(0.5)$ of the counter-monotonic sum is equal to 0.33610. The first term gives:

$$\pi_{X_2}\left(F_{X_2}^{-1}(1 - u_{x,1})\right) - \lambda_{X_1}\left(F_{X_1}^{-1}(u_{x,1})\right) = 0.03918,$$

whereas $\mathcal{S}_{x,12} = 0.92823$. The remaining part due to jumps gives:

$$u_{x,1}\left(g(u_{x,1}) - x\right) + \mathcal{J}_{x,12} = -0.63130.$$

This shows the importance of the last term, as ignoring it would considerably overestimate the value of the stop-loss premium.

7 Conclusion

This paper studies the Value-at-Risk, the Tail Value-at-Risk, and the upper tail transform of the sum of two counter-monotonic random variables $S^l \stackrel{d}{=} F_{X_1}^{-1}(U) + F_{X_2}^{-1}(1 - U)$. Decomposition formulas for these risk measures are derived in terms of the corresponding risk measures of the marginal random variables. An important step in the derivation is to study the behavior of the function $g : u \mapsto F_{X_1}^{-1}(u) + F_{X_2}^{-1}(1 - u)$. This is performed by introducing the set E_x which allows to identify the crossing points of g with respect to a level x.

The derivations of this paper do not require any strong restrictions on the random variables X_1 and X_2. In particular, these random variables can be continuous, discrete, or a combination of the two. The results are also valid for random variables which are defined on the entire real support, and are not limited to positive valued random variables.

The contributions of this paper are relevant to different areas of finance and actuarial science. One of the applications is asset and liability management where differences of random variables

are involved rather than sums. In this case, the counter-monotonic sum becomes a comonotonic difference. The study of basis risk is also an area of application where the relevant quantity is a difference of two random variables.

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A   Appendix : proofs

A.1   Proof of Lemma 4.1

For $u_{x,j} \in E_x$, when the function g is continuous in $u_{x,j}$, the equality $g(u_{x,j}-) = g(u_{x,j}+) = x$ holds, and hence:

$$F_{X_1}^{-1(\alpha_{x,j})}(u_{x,j}) + F_{X_2}^{-1(1-\alpha_{x,j})}(1 - u_{x,j}) = F_{X_1}^{-1}(u_{x,j}) + F_{X_2}^{-1}(1 - u_{x,j}) = x,$$

for any $\alpha_{x,j} \in [0,1]$. Thus, Equation (4.2) holds in particular for $\alpha_{x,j} = 0$.

When the function g jumps in $u_{x,j}$, the jump size is $|g(u_{x,j}+) - g(u_{x,j}-)|$. Then, there exists $\alpha_{x,j} \in [0,1]$ such that:

$$x = (1 - \alpha_{x,j})\, g(u_{x,j}-) + \alpha_{x,j} g(u_{x,j}+). \tag{A.1}$$

Note that $F_{X_2}^{-1}(1 - (u_{x,j}-)) = F_{X_2}^{-1}(\tilde{u}_{x,j}+)$, and $F_{X_2}^{-1}(1 - (u_{x,j}+)) = F_{X_2}^{-1}(\tilde{u}_{x,j}-)$, where $\tilde{u}_{x,j} = 1 - u_{x,j}$. Moreover, the inverses $F_{X_i}^{-1}$ are left-continuous, which means that $F_{X_i}^{-1}(q-) = F_{X_i}^{-1}(q)$ and $F_{X_i}^{-1}(q+) = F_{X_i}^{-1+}(q)$ for any $q \in (0,1)$. Therefore, using (2.1), it follows that (A.1) is equivalent with:

$$x = F_{X_1}^{-1(\alpha_{x,j})}(u_{x,j}) + F_{X_2}^{-1(1-\alpha_{x,j})}(1 - u_{x,j}).$$

Using again (A.1) and rearranging leads to Expression (4.3) of $\alpha_{x,j}$ when g has a jump in $u_{x,j}$.

A.2   Proof of Theorem 5.1

Using the notation $x_p^\alpha = F_{S^l}^{-1(\alpha)}(p)$, where $x_p^\alpha \in (x^{\min}, x^{\max})$ for $p \in (0,1)$, the TVaR of the counter-monotonic sum can be written using (2.7) as follows:

$$\mathrm{TVaR}_p\left[S^l\right] = \frac{1}{1-p}\left(\int_0^1 \left(F_{S^l}^{-1}(u) - x_p^\alpha\right)_+ \mathrm{d}u + x_p^\alpha\, (1-p)\right). \tag{A.2}$$

Since $S^l \overset{d}{=} g(U)$, it holds that:

$$\int_0^1 \left(F_{S^l}^{-1}(u) - x_p^\alpha\right)_+ du = \int_0^1 \left(g(u) - x_p^\alpha\right)_+ du. \tag{A.3}$$

Let $u_{p,j}^\alpha$ for $j = 1, ..., N_p^\alpha$ be the N_p^α ordered elements of the set $E_{x_p^\alpha}$. Suppose first that N_p^α is finite. The following holds:

$$\int_0^1 \left(g(u) - x_p^\alpha\right)_+ du = \int_0^{u_{p,1}^\alpha} \left(g(u) - x_p^\alpha\right)_+ du + \sum_{j=1}^{N_p^\alpha - 1} \int_{u_{p,j}^\alpha}^{u_{p,j+1}^\alpha} \left(g(u) - x_p^\alpha\right)_+ du + \int_{u_{p,N_p^\alpha}^\alpha}^1 \left(g(u) - x_p^\alpha\right)_+ du.$$

The sign of the function $g(u) - x_p^\alpha$ alternates over two consecutive intervals $\left(u_{p,j-1}^\alpha, u_{p,j}^\alpha\right)$ and $\left(u_{p,j}^\alpha, u_{p,j+1}^\alpha\right)$, for $j = 1, ..., N_p^\alpha$, and the sign of $g(u) - x_p^\alpha$ on $(0, u_{p,1}^\alpha)$ determines the sign of $g(u) - x_p^\alpha$ on all subsequent intervals. Therefore, depending on whether N_p^α is either even or odd, and whether the sign of $g(u) - x_p^\alpha$ is either positive or negative on $\left(0, u_{p,1}^\alpha\right)$, four possible cases can be distinguished.

The first case is when N_p^α is an even number (i.e. there exists an integer m such that $N_p^\alpha = 2m$), and $g(u) \geq x_p^\alpha$ for $u \in \left(0, u_{p,1}^\alpha\right)$. In this case, it follows that:

$$\int_0^1 \left(g(u) - x_p^\alpha\right)_+ du = \int_0^{u_{p,1}^\alpha} \left(g(u) - x_p^\alpha\right) du + \sum_{j=1}^{m-1} \int_{u_{p,2j}^\alpha}^{u_{p,2j+1}^\alpha} \left(g(u) - x_p^\alpha\right) du + \int_{u_{p,N_p^\alpha}^\alpha}^1 \left(g(u) - x_p^\alpha\right) du. \tag{A.4}$$

The second case is when N_p^α is an even number (i.e. $N_p^\alpha = 2m$), and $g(u) \leq x_p^\alpha$ for $u \in (0, u_{p,1}^\alpha)$, which leads to:

$$\int_0^1 \left(g(u) - x_p^\alpha\right)_+ du = \sum_{j=1}^m \int_{u_{p,2j-1}^\alpha}^{u_{p,2j}^\alpha} \left(g(u) - x_p^\alpha\right) du. \tag{A.5}$$

The third case is when N_p^α is an odd number (i.e. there exists an integer m such that $N_p^\alpha = 2m + 1$), and $g(u) \geq x_p^\alpha$ for $u \in (0, u_{p,1}^\alpha)$. In this case, it follows that:

$$\int_0^1 \left(g(u) - x_p^\alpha\right)_+ du = \int_0^{u_{p,1}^\alpha} \left(g(u) - x_p^\alpha\right) du + \sum_{j=1}^m \int_{u_{p,2j}^\alpha}^{u_{p,2j+1}^\alpha} \left(g(u) - x_p^\alpha\right) du. \tag{A.6}$$

The fourth case is when N_p^α is an odd number (i.e. $N_p^\alpha = 2m + 1$), and $g(u) \leq x_p^\alpha$ for $u \in (0, u_{p,1}^\alpha)$, which leads to:

$$\int_0^1 \left(g(u) - x_p^\alpha\right)_+ du = \sum_{j=1}^m \int_{u_{p,2j-1}^\alpha}^{u_{p,2j}^\alpha} \left(g(u) - x_p^\alpha\right) du + \int_{u_{p,N_p^\alpha}^\alpha}^1 \left(g(u) - x_p^\alpha\right) du. \tag{A.7}$$

For $j = 1, ..., N_p^\alpha - 1$, the following integral split holds:

$$\int_{u_{p,j}}^{u_{p,j+1}} (g(u) - x_p) \, \mathrm{d}u = \int_{u_{p,j}}^{1} (g(u) - x_p) \, \mathrm{d}u - \int_{u_{p,j+1}}^{1} (g(u) - x_p) \, \mathrm{d}u. \qquad \text{(A.8)}$$

Taking (A.8) into account in (A.4)-(A.7), the four cases reduce to two only, for any N_p^α. Namely:

$$\int_0^1 \left(F_{S^l}^{-1}(u) - x_p^\alpha \right)_+ \mathrm{d}u = \begin{cases} I_1^\alpha(p), & \text{if } g(u) \le x_p^\alpha \text{ for } u \in (0, u_{p,1}^\alpha), \\ I_2^\alpha(p), & \text{if } g(u) \ge x_p^\alpha \text{ for } u \in (0, u_{p,1}^\alpha), \end{cases} \qquad \text{(A.9)}$$

where

$$I_1^\alpha(p) = \int_{u_{p,1}^\alpha}^1 \left(g(u) - x_p^\alpha \right) \mathrm{d}u - \sum_{j=2}^{N_p^\alpha} (-1)^j \int_{u_{p,j}^\alpha}^1 \left(g(u) - x_p^\alpha \right) \mathrm{d}u, \qquad \text{(A.10)}$$

$$I_2^\alpha(p) = \int_0^{u_{p,1}^\alpha} \left(g(u) - x_p^\alpha \right) \mathrm{d}u + \sum_{j=2}^{N_p^\alpha} (-1)^j \int_{u_{p,j}^\alpha}^1 \left(g(u) - x_p^\alpha \right) \mathrm{d}u. \qquad \text{(A.11)}$$

Taking into account the definitions of the TVaR and LTVaR from (2.2) and (2.3), respectively, and using the appropriate change of variable for the integral over $F_{X_2}^{-1}(1 - u)$, it follows that:

$$\int_{u_{p,j}^\alpha}^1 \left(g(u) - x_p^\alpha \right) \mathrm{d}u = \left(1 - u_{p,j}^\alpha \right) \left(\mathrm{TVaR}_{u_{p,j}^\alpha}[X_1] + \mathrm{LTVaR}_{1-u_{p,j}^\alpha}[X_2] \right) - x_p^\alpha \left(1 - u_{p,j}^\alpha \right), \qquad \text{(A.12)}$$

for all $j = 1, ..., N_p^\alpha$, as well as:

$$\int_0^{u_{p,1}^\alpha} \left(g(u) - x_p^\alpha \right) \mathrm{d}u = u_{p,1}^\alpha \left(\mathrm{LTVaR}_{u_{p,1}^\alpha}[X_1] + \mathrm{TVaR}_{1-u_{p,1}^\alpha}[X_2] \right) - x_p^\alpha u_{p,1}^\alpha. \qquad \text{(A.13)}$$

Plugging (A.12) and (A.13) in (A.10) and (A.11), the expressions of $I_1^\alpha(p)$ and $I_2^\alpha(p)$ can be written as follows:

$$I_1^\alpha(p) = \left(1 - u_{p,1}^\alpha \right) \left(\mathrm{TVaR}_{u_{p,1}^\alpha}[X_1] + \mathrm{LTVaR}_{1-u_{p,1}^\alpha}[X_2] \right) - x_p^\alpha \left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha \right) - \mathcal{T}_{p,N_p^\alpha}^\alpha,$$

$$I_2^\alpha(p) = u_{p,1}^\alpha \left(\mathrm{LTVaR}_{u_{p,1}^\alpha}[X_1] + \mathrm{TVaR}_{1-u_{p,1}^\alpha}[X_2] \right) - x_p^\alpha \left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha \right) + \mathcal{T}_{p,N_p^\alpha}^\alpha,$$

where $\mathcal{T}_{p,N_p^\alpha}^\alpha$ and $\mathcal{D}_{p,N_p^\alpha}^\alpha$ are defined in (5.3). Using (A.2), the TVaR of the counter-comonotonic sum is given by

$$(1-p)\mathrm{TVaR}_p\left[S^l\right] = \begin{cases} t_1^\alpha(p), & \text{if } g(u) \le x_p^\alpha \text{ for } u \in (0, u_{p,1}^\alpha), \\ t_2^\alpha(p), & \text{if } g(u) \ge x_p^\alpha \text{ for } u \in (0, u_{p,1}^\alpha), \end{cases}$$

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where:

$$t_1^\alpha(p) = I_1^\alpha(p) + x_p^\alpha(1-p),$$
$$t_2^\alpha(p) = I_2^\alpha(p) + x_p^\alpha(1-p).$$

The proof ends by noting that combining $-x_p^\alpha\left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha\right)$ and $x_p^\alpha(1-p)$ leads to:

$$x_p^\alpha\left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right),$$

whereas combining $-x_p^\alpha\left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha\right)$ and $x_p^\alpha(1-p)$ leads to:

$$x_p^\alpha\left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right).$$

A.3   Proof of Lemma 5.1

For any $p \in (0,1)$ and any $\alpha \in [0,1]$, the quantile $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p if and only if the cdf F_{S^l} is continuous in $x_p^\alpha = F_{S^l}^{-1(\alpha)}(p)$. In this case, $F_{S^l}\left(x_p^\alpha\right) = p$, and hence, the following equivalence holds:

$$q \le p \Leftrightarrow F_{S^l}^{-1(\alpha)}(q) \le F_{S^l}^{-1(\alpha)}(p), \qquad \text{for } q \in (0,1).$$

Combining this equivalence with the fact that $S^l \overset{d}{=} g(U)$ and $S^l \overset{d}{=} F_{S^l}^{-1(\alpha)}(U)$ for any $\alpha \in [0,1]$, the TVaR can be written as follows:

$$\text{TVaR}_p\left[S^l\right] = \mathbb{E}\left[g(U)\mathbb{I}\left[g(U) \ge x_p^\alpha\right]\right] = \frac{1}{1-p}\int_0^1 g(u)\mathbb{I}\left[g(u) \ge x_p^\alpha\right]\mathrm{d}u. \qquad (A.14)$$

Further, for $p \in (0,1)$, the function $F_{S^l}^{-1(\alpha)}$ is strictly increasing in p if and only if:

$$\mathbb{P}\left[S^l = x_p^\alpha\right] = \mathbb{P}\left[g(U) = x_p^\alpha\right] = 0,$$

and hence:

$$\mathbb{P}\left[g(U) \ge x_p^\alpha\right] = \mathbb{P}\left[g(U) > x_p^\alpha\right].$$

Therefore, the integral in (A.14) can be decomposed as in (A.7), where the following holds:

$$\int_{u_{p,j}^\alpha}^1 g(u)\mathrm{d}u = \left(1 - u_{p,j}^\alpha\right)\left(\text{TVaR}_{u_{p,j}^\alpha}[X_1] + \text{LTVaR}_{1-u_{p,j}^\alpha}[X_2]\right),$$

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for all $j = 1, ..., N_p^\alpha$, where $u_{p,j}^\alpha$ are the N_p^α elements of $E_{x_p^\alpha}$, as well as:

$$\int_0^{u_{p,1}^\alpha} g(u)\mathrm{d}u = u_{p,1}^\alpha \left(\mathrm{LTVaR}_{u_{p,1}^\alpha}[X_1] + \mathrm{TVaR}_{1-u_{p,1}^\alpha}[X_2]\right).$$

In particular, the terms $F_{Sl}^{-1(\alpha)}(p)\left(u_{p,1}^\alpha + \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right)$ and $F_{Sl}^{-1(\alpha)}(p)\left(1 - u_{p,1}^\alpha - \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right)$ in $t_1^\alpha(p)$ and $t_2^\alpha(p)$, respectively, are equal to 0, which proves (5.4).

A.4 Proof of Theorem 6.1

For $x \in \left(x^{\min}, x^{\max}\right)$, there always exist $p \in (0,1)$ and $\alpha \in [0,1]$ such that $x = F_{Sl}^{-1(\alpha)}(p)$. Let $u_{x,1}, ..., u_{x,N_x}$ be the N_x elements of the set E_x. Note that $u_{x,j} = u_{p,j}^\alpha$ for all $j = 1, ..., N_x$, and $N_x = N_p^\alpha$. Using the relationships (2.7) and (2.8), it follows that:

$$
\begin{aligned}
(1 - u_{x,j})\left(\mathrm{TVaR}_{u_{x,j}}[X_1] + \mathrm{LTVaR}_{1-u_{x,j}}[X_2]\right) &= (1 - u_{x,j})g(u_{x,j}) \quad\quad\quad\quad\text{(A.15)} \\
&+ \pi_{X_1}\left(F_{X_1}^{-1}(u_{x,j})\right) - \lambda_{X_2}\left(F_{X_2}^{-1}(1 - u_{x,j})\right).
\end{aligned}
$$

Define the quantity A as follows:

$$A = (1 - u_{x,1})\left(\mathrm{TVaR}_{u_{x,1}}[X_1] + \mathrm{LTVaR}_{1-u_{x,1}}[X_2]\right) - \mathcal{T}_{p,N_p^\alpha}^\alpha,$$

where $\mathcal{T}_{p,N_p^\alpha}^\alpha$ is given by (5.3). Multiplying Expression (A.15) by $(-1)^j$ and summing over j leads to:

$$
\begin{aligned}
A &= \pi_{X_1}\left(F_{X_1}^{-1}(u_{x,1})\right) - \lambda_{X_2}\left(F_{X_2}^{-1}(1 - u_{x,1})\right) - \mathcal{S}_{x,N_x} \\
&\quad + (1 - u_{x,1})g(u_{x,1}) - \sum_{j=2}^{N_x}(-1)^j\,(1 - u_{x,j})\,g(u_{x,j}),
\end{aligned}
$$

where $\mathcal{S}_{x,N_x}$ is given by (6.3). By adding $x\left(u_{x,1} + \mathcal{D}_{p,N_p^\alpha}^\alpha - p\right)$ to the left-hand side, where $\mathcal{D}_{p,N_p^\alpha}^\alpha$ is defined in (5.3), it follows that:

$$
\begin{aligned}
t_1^\alpha(p) &= \pi_{X_1}\left(F_{X_1}^{-1}(u_{x,j})\right) - \lambda_{X_2}\left(F_{X_2}^{-1}(1 - u_{x,j})\right) - \mathcal{S}_{x,N_x} + (1 - u_{x,1})g(u_{x,1}) \\
&\quad - \sum_{j=2}^{N_x}(-1)^j\,(1 - u_{x,j})\,g(u_{x,j}) + x\left(u_{x,1} - 1 + \mathcal{D}_{p,N_p^\alpha}^\alpha + (1 - p)\right),
\end{aligned}
$$

which can be rearranged such that $t_1^\alpha(p) = s_1(x) + (1-p)x$. The same reasoning allows to express $t_2^\alpha(p)$ in terms of $s_2(x)$. Therefore, it follows that:

$$
\begin{aligned}
t_1^\alpha(p) &= s_1(x) + (1-p)x, \\
t_2^\alpha(p) &= s_2(x) + (1-p)x,
\end{aligned}
\tag{A.16}
$$

for any $x \in \left(x^{\min}, x^{\max}\right)$. Note also that (2.7) immediately leads to:

$$
\pi_{S^l}(x) = (1-p)\,\text{TVaR}_p\left[S^l\right] - (1-p)\,x,
\tag{A.17}
$$

for any $x \in \left(x^{\min}, x^{\max}\right)$. If $g(u) \le x$ on $(0, u_{x,1})$, then $\text{TVaR}_p\left[S^l\right] = \frac{1}{1-p}t_1^\alpha(p)$, and combining (A.16) and (A.17) proves that $\pi_{S^l}(x) = s_1(x)$. Similarly, $\pi_{S^l}(x) = s_2(x)$ if $g(u) \ge x$ on $(0, u_{x,1})$.

A.5 Proof of Theorem 6.2

The first part of the proof shows that Expressions (6.2) and (6.5) of $s_1(x)$ are equal.

For $j = 1, ..., N_x$, let $\alpha_{x,j} \in [0,1]$ be such that:

$$
F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right) + F_{X_2}^{-1(1-\alpha_{x,j})}\left(1 - u_{x,j}\right) = x,
\tag{A.18}
$$

for $x \in (x^{\min}, x^{\max})$, where $\alpha_{x,j}$ is known to exist from Lemma 4.1. The stop-loss premium of X_1 with retention $F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)$ can be expressed as follows:

$$
\pi_{X_1}\left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)\right) = \int_{F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)}^{+\infty} \left(1 - F_{X_1}(y)\right)\mathrm{d}y.
\tag{A.19}
$$

For any $\alpha_{x,j} \in [0,1]$, the following equality holds:

$$
F_{X_1}\left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)\right) = F_{X_1}\left(F_{X_1}^{-1}\left(u_{x,j}\right)\right) = u_{x,j}.
\tag{A.20}
$$

Combining (A.19) and (A.20) allows to link the stop-loss premiums of X_1 with retentions $F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)$ and $F_{X_1}^{-1}\left(u_{x,j}\right)$ as follows:

$$
\pi_{X_1}\left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)\right) = \pi_{X_1}\left(F_{X_1}^{-1}\left(u_{x,j}\right)\right) - \left(F_{X_1}^{-1(\alpha_x)}\left(u_{x,j}\right) - F_{X_1}^{-1}\left(u_{x,j}\right)\right)\left(1 - u_{x,j}\right),
\tag{A.21}
$$

Similarly, the lower-tail transform of X_2 with retention $F_{X_2}^{-1(1-\alpha_{x,j})}\left(1 - u_{x,j}\right)$ has the following

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expression:

$$\lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)\right) = \int_{-\infty}^{F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)} F_{X_2}(y)\mathrm{d}y, \tag{A.22}$$

and since the following equality holds:

$$F_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)\right) = F_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,j}\right)\right) = 1 - u_{x,j}, . \tag{A.23}$$

for any $\alpha_{x,j} \in [0,1]$, the lower-tail transforms of X_2 with retentions $F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)$ and $F_{X_2}^{-1}\left(1-u_{x,j}\right)$ can be linked as follows:

$$\begin{aligned}\lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)\right) &= \lambda_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,j}\right)\right)\\ &\quad + \left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right) - F_{X_2}^{-1}\left(1-u_{x,j}\right)\right)\left(1-u_{x,j}\right).\end{aligned} \tag{A.24}$$

For $j = 1, ..., N_x$, consider the following difference:

$$A_j = \pi_{X_1}\left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)\right).$$

Combining Expressions (A.21) and (A.24) leads to:

$$\begin{aligned}A_j &= \pi_{X_1}\left(F_{X_1}^{-1}\left(u_{x,j}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,j}\right)\right) - \left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right) - F_{X_1}^{-1}\left(u_{x,j}\right)\right)\left(1-u_{x,j}\right)\\ &\quad - \left(F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right) - F_{X_2}^{-1}\left(1-u_{x,j}\right)\right)\left(1-u_{x,j}\right)\\ &= \pi_{X_1}\left(F_{X_1}^{-1}\left(u_{x,j}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,j}\right)\right) +\\ &\quad \left(1-u_{x,j}\right)\left[\left(F_{X_1}^{-1}(u_{x,j}) + F_{X_2}^{-1}\left(1-u_{x,j}\right)\right) - \left(F_{X_1}^{-1(\alpha_{x,j})}\left(u_{x,j}\right) + F_{X_2}^{-1(1-\alpha_{x,j})}\left(1-u_{x,j}\right)\right)\right]\\ &= \pi_{X_1}\left(F_{X_1}^{-1}\left(u_{x,j}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,j}\right)\right) + \left(1-u_{x,j}\right)\left(g(u_{x,j}) - x\right), \tag{A.25}\end{aligned}$$

where both the definition of the function g and the relation (A.18) were used in the last step. Therefore, (A.25) proves that the expressions of $s_1(x)$ given in (6.2) and (6.5) are equal.

It remains to prove that Expressions (6.2) and (6.5) of $s_2(x)$ are equal. Define:

$$B = \pi_{X_2}\left(F_{X_2}^{-1(\alpha_{x,1})}\left(u_{x,1}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1(1-\alpha_{x,1})}\left(1-u_{x,1}\right)\right).$$

Using the same arguments as above, it follows that:

$$B = \pi_{X_2}\left(F_{X_2}^{-1}\left(u_{x,1}\right)\right) - \lambda_{X_2}\left(F_{X_2}^{-1}\left(1-u_{x,1}\right)\right) + u_{x,1}\left(g(u_{x,1}) - x\right), \tag{A.26}$$

which can be combined with (A.25) for $j = 2, ..., N_x$ to prove that (6.2) and (6.5) are two equivalent expressions for $s_2(x)$ as well, which ends the proof.

A.6   Proof of Lemma 6.1

First, note that using the parity (2.6), it follows that:

$$s_1(x) + s_2(x) = \mathbb{E}\left[S^l\right] - x.$$

Suppose that $\pi_{S^l}(x) = s_1(x)$. Then, from (2.6), $\lambda_{S^l}(x) = -s_2(x)$. Since the lower tail transform is always non-negative, it follows that $s_2(x) \leq 0$, and hence, the non-negativeness of the upper tail transform leads to $s_2(x) \leq s_1(x)$. The same reasoning applies when $\pi_{S^l}(x) = s_2(x)$, in which case $s_2(x) \geq s_1(x)$. This concludes the proof for the stop-loss premium.

For $p \in (0,1)$ and $x_p^\alpha = F_{S^l}^{-1}(p)$, the first part of the proof leads to:

$$\pi_{S^l}\left(x_p^\alpha\right) = \max\left\{s_1\left(x_p^\alpha\right), s_2\left(x_p^\alpha\right)\right\}. \tag{A.27}$$

Plugging (A.16) and (A.17) in (A.27) proves that the TVaR of S^l is indeed the maximum of $\frac{1}{1-p}t_1^\alpha(p)$ and $\frac{1}{1-p}t_2^\alpha(p)$, and hence ends the proof.

A.7   Proof of Corollary 2

For $N_x = 1$, the set E_x contains a single element $u_{x,1}$ and $\mathcal{S}_{x,1} = 0$, and Theorem 6.2 leads to a simplified decomposition. The remaining task is to prove that the retentions of the upper and lower tail transforms can be expressed in terms of the cdf of S^l.

Note that $S^l \overset{d}{=} g(U)$ leads to:

$$F_{S^l}(x) = \mathbb{P}\left[g(U) \leq x\right]. \tag{A.28}$$

In the first case where $g(u) \leq x$ for $u < u_{x,1}$, it follows immediately from (A.28) that:

$$F_{S^l}(x) = u_{x,1}. \tag{A.29}$$

In the second case where $g(u) \geq x$ for $u < u_{x,1}$, unlike in the first case, the following inequality holds:

$$F_{S^l}(x) \geq \mathbb{P}\left[g(U) < x\right] = 1 - u_{x,1}. \tag{A.30}$$

Equality in (A.30) is satisfied if g is strictly decreasing in $u_{x,1}$, in which case $F_{S^l}(x) = 1 - u_{x,1}$. Otherwise, from the definition of the set E_x, the case where the function g is flat before $u_{x,1}$ leads to an inequality, and there exists $u^\star < u_{x,1}$ such that $g(u) = x$ for $u \in (u^\star, u_{x,1})$. Thus, $F_{S^l}(x) = 1 - u^\star$, and the upper tail transform of S^l can be written as follows:

$$\mathbb{E}\left[\left(S^l - x\right)_+\right] = \int_0^{u^\star} (g(u) - x)\, \mathrm{d}u. \tag{A.31}$$

Using Lemma 4.1, there exists $\alpha_x \in [0,1]$ such that:

$$x = F_{X_1}^{-1(\alpha_x)}(u^\star) + F_{X_2}^{-1(1-\alpha_x)}(1 - u^\star) = F_{X_1}^{-1(\alpha_x)}(1 - F_{S^l}(x)) + F_{X_2}^{-1(1-\alpha_x)}(F_{S^l}(x)). \tag{A.32}$$

Combining (A.31) and (A.32) and rearranging ends the proof.