The 3-step hedge-based valuation: fair valuation in the presence of systematic risks

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Abstract

In this paper we introduce the 3-step hedge-based valuation for the valuation of general hybrid claims which depend on financial, actuarial and systematic risks. Our valuation is a market-consistent and model-consistent valuation that takes into account that the hedgeable, actuarial and systematic part of a claim have to be valuated with different valuation principles. We also consider the additive 3-step valuation which was introduced in Deelstra et al. (2020) and show that this additive valuation is a member of the more general class of 3-step hedge-based valuations. We illustrate our new fair valuation using intuitive examples and compare the results with the hedge-based valuation introduced in Dhaene et al. (2017) and the 2-step valuation introduced in Pelsser and Stadje (2014).

Keywords: market-consistent valuation, actuarial valuation, systematic risk, fair valuation.

1 Introduction

The value of a random future liability should correspond with the amount of money required today to set up an appropriate risk management strategy and to ensure a sufficiently large likelihood to meet the future liability. Insurance liabilities are complex combinations of different types of risks. Each type requires a different risk management strategy and therefore a different valuation principle. In this paper we determine a market-consistent, 3-step valuation that decomposes the claim into a hedgeable, systematic and diversifiable part and determines the value of the claim by combining appropriate financial, systematic and actuarial valuation principles.

A diversifiable claim is a combination of individual risks, each of them having its own specific characterics. Therefore, one may benefit from risk reducing diversification effects when pooling individual risks. The law of large numbers shows that a diversifiable claim eventually becomes deterministic if a large amount of individual risks are aggregated. The valuation of the claim should therefore be based on the expected loss. A risk margin can be added to reflect the imperfect diversification in a finite portfolio and to absorb unexpected deviations from the expectation.

Traditional insurance liabilities in life and non-life insurance are often assumed to be diversifiable. For example, the underlying assumptions when determining pure premiums for life insurance and annuity benefits is that the insurer will aggregate a large amount of independent policies; see e.g. Dickson et al. (2019). Non-life insurers employ statistical methods to determine rating variables which are then used to classify policies in homogeneous groups; see e.g Brockman and Wright (1992), Denuit and Lang (2004). An overview of different actuarial premium principles can be found in Denuit et al. (2006) and Kaas et al. (2008).

A hedgeable claim is a liability that can be managed by investing in an appropriate linear combination of traded assets. For example, stocks, bonds and other traded derivatives are examples of traded assets that can be used to replicate the payoff of a future liability. If such a replicating portfolio exists, buying the replicating portfolio will eliminate the risks of the liability. The claim is 'hedged' and the value of such a hedgeable claim should correspond with the market value of the replicating portfolio to avoid arbitrage opportunities. The value of the replicating portfolio can be observed in the market and therefore valuation of a hedgeable claim is model free.

The payout of a financial claim, or derivative, is a function of the traded assets. Financial claims are not necessarily hedgeable. Their pricing is based on no-arbitrage considerations. It was shown in Delbaen and Schachermayer (2006) that the arbitrage-free price of a financial claim corresponds with a discounted expectation under a risk neutral probability measure. For a non-hedgeable claim, the choice of the risk neutral probability measure, and therefore also the price of the claim, is not unique. In Frittelli (2000) and Dhaene et al. (2015), the authors consider financial pricing using the minimal entropy martingale measure. Alternatively, one can use the stochastic discount factor approach (see Hansen and Jagannathan (1997) and van Bilsen and Linders (2019)) or use actuarial risk measures to select a risk neutral probability measure; see e.g. Bühlmann et al. (1996), Gerber and Shiu (1995), Hubalek and Sgarra (2006), Goovaerts and Laeven (2008), Pelsser (2008).

A systematic claim is a claim which is not diversifiable nor hedgeable. Therefore, hedging and diversification cannot be used to manage the risks of a systematic claim. Indeed, a systematic risk is not traded, which means financial markets will not be able to offset the systematic risk. Moreover, risks driven by systematic risk factors exhibit a strong positive dependence. Therefore, aggregating systematic risks does not lead to the desired risk reduction. The valuation of a systematic claim should be different from the actuarial and financial valuations and consider a prudent approach. For example, one can determine appropriate risk margins by determining the expected systematic loss under a stressed scenario, i.e. consider a situation which is 'less preferable' than the real situation; see e.g. Börger (2010), Zeddouk and Devolder (2019).

In this paper we propose a 3-step hedge-based valuation principle for hybrid claims. A hybrid claim depends on diversifiable, hedgeable and systematic risk factors. Therefore, we combine the replicating portfolio approach, traditional actuarial valuation as well as systematic valuation principles. In a first step, we determine an appropriate hedging portfolio to offset the hybrid liability. A hybrid liability depends on non-traded risks and therefore a hedging portfolio will not be capable of exactly replicating the hybrid liability. The residual part of the claim corresponds with the hybrid liability after subtracting the random income generated by the hedging portfolio. In the second step, we use a conditional actuarial valuation to determine the value of the non-hedgeable part of the claim. The conditional actuarial valuation delivers a random variable which represents the actuarial value of the non-hedgeable claim for each realization of the financial and systematic risks. Therefore, after applying the conditional actuarial valuation, we are left with a claim that depends on the traded and systematic risks. The third valuation step then consists of pricing this non-actuarial claim using a systematic valuation.

The 3-step hedge-based valuation is a combination of the hedge-based valuation introduced in Dhaene et al. (2017) and the 2-step valuation introduced in Pelsser and Stadje (2014). Both valuations assume a market with actuarial and traded risks and decompose a hybrid claim in a financial and an actuarial part. These valuations are therefore a combination of financial and actuarial pricing principles. We will show that our 3-step hedge-based valuation is consistent with the hedge-based valuation and the 2-step valuation. Indeed, in the presence of systematic risks, the hedge-based and the 2-step valuation will consider these systematic risks as non hedgeable and therefore use the actuarial valuation to value the systematic part of the claim; see e.g. Dhaene (2020). However, the 3-step hedge-based valuation allows to use a valuation principle which is different from the actuarial valuation to manage the systematic part of the hybrid claim.

The idea of decomposing a hybrid claim in a hedgeable, actuarial and systematic part was also proposed in Dhaene (2020) and Deelstra et al. (2020). In both situations, the authors consider a product claim, i.e. the hybrid claim is the product of a hedgeable claim and an actuarial claim. Moreover, the actuarial risks and financial risks are assumed to be independent. We generalize these approaches in that the 3-step hedge-based valuation allows for general hybrid claims and dependencies between the financial and actuarial risks. In Deelstra et al. (2020), the authors use a different 3-step valuation principle. Indeed, after decomposing the hybrid claim into three parts, the valuation of the hybrid claim is assumed to be the sum of the valuations of the different parts. We show in this paper that such an additive 3-step valuation is a subset of the larger class of 3-step hedge-based valuations.

The 3-step hedge-based valuation is a market-consistent valuation. Insurance regulations, such as Solvency II, often require that insurance liabilities are valued using the available observable market prices in a deep and liquid market (mark-to-market), whereas actuarial judgement is used to value the remaining part (mark-to-model); see e.g. Möhr (2011). This leads to the notion of a fair valuation. Fair valuations for unhedgeable claims were already considered in Neuberger and Hodges (1989) using a utility indifference approach. In Malamud et al. (2008) the utility indifference framework is used to define market-consistent valuations for insurance claims. Market-consistent valuation requires a combination of actuarial and financial valuation methods, as was first pointed out by Brennan and Schwartz (1976), who considered the valuation of guarantees in unit-linked insurance contracts; see also Embrechts (2000). Recent approaches to define market-consistent valuations are: Pelsser and Stadje (2014), Pelsser and Schweizer (2016), Wüthrich (2016), Dhaene et al. (2017), Delong et al. (2019*a*), Barigou and Dhaene (2019), Barigou et al. (2019, 2020), Deelstra et al. (2020), Bacinello et al. (2021), Chen et al. (2021).

The contributions of this paper are as follows. First, we introduce a new market-consistent valuation which allows to deal with systematic risks. Second, we consider a characterization of the

3-step hedge-based valuation for a linear systematic valuation. Third, we consider an additive 3-step valuation and show that this valuation is a special case of the 3-step hedge-based valuations. Last, we derive closed-form expressions for product claims consisting of a hedgeable financial claim and an insurance portfolio with conditional independent risks. Moreover, we compare the 3-step valuation with existing market-consistent valuations, such as the hedge-based and the 2-step valuation.

This paper is organized as follows. In Section 2 we introduce the financial, actuarial and systematic claims and their valuations. Moreover, we define a hybrid claim as a claim consisting of at least two different type of risks. The notion of market-consistency is discussed in Section 3 as well as the hedge-based valuation and the 2-step valuation. We propose our 3-step hedge-based valuation in Section 4, together with the important properties and an illustration. In Section 5 we introduce the additive 3-step valuation and show that this valuation is a special case of the more general class of 3-step hedge-based valuations. Moreover, we consider an example and compare the 3-step valuation with the 2-step, the hedge-based valuation and the 3-step valuation introduced in Deelstra et al. (2020).

2 Hybrid claims

A claim is a future cashflow which has to be paid by the insurance company to the policyholders. In this paper we assume claims to be random variables depending on different types of risk drivers: financial, systematic and diversifiable risk drivers. We assume today is time $t = 0$ and the claim has a deterministic maturity $T > 0$. The aim is to determine the time-0 value of such a future liability. We assume that all random variables encountered in this paper are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and have finite first two moments. The set of all random variables is denoted by $\mathcal{C}$.

2.1 Valuations

Definition 2.1 (Valuation) *A valuation is a mapping $\rho : \mathcal{C} \to \mathbb{R}$ attaching the real number $\rho[S]$ to any claim $S \in \mathcal{C}$ such that*

1. *ρ is normalized: $\rho[0] = 0$,*
2. *ρ is translation invariant: $\rho[S + a] = a + \rho[S]$, for $a \in \mathbb{R}$.*

The real number $\rho[S]$ can be interpreted as the value of a claim S. A special class of valuations are the linear valuations, which are defined below.

Definition 2.2 (Linear valuation) *A valuation $\rho : \mathcal{C} \to \mathbb{R}$ is said to be linear if there exists a non-negative random variable φ such that:*

$$\rho[S] = \mathbb{E}[\varphi S],$$

for all $S \in \mathcal{C}$.

A valuation is by definition normalized and translation invariant. However, we often work with valuations that possess additional useful properties. For example, we may have that the valuation is monotone, homogeneous or subadditive:

1. Monotone: if $S_1 \leq S_2$, then $\rho[S_1] \leq \rho[S_2]$.
2. Positive homogeneous: $\rho[aS] = a\rho[S],$ for $a \geq 0$.
3. Subadditive: $\rho[S_1 + S_2] \leq \rho[S_1] + \rho[S_2]$.

A linear valuation principle is positive homogeneous, monotone and subadditive. Moreover, a linear valuation ρ is an additive valuation principle: $\rho[S_1 + S_2] = \rho[S_1] + \rho[S_2]$. If we consider valuations which are monotone, positive homogeneous and subadditive, but not necessarily additive, we consider a larger set of valuations, which are labeled the coherent valuations; see Artzner et al. (1999).

2.2 Financial claims

We assume there is an arbitrage-free financial market and payoffs with maturity T are traded in this market. These traded assets can be bought and sold by all market participants at any quantity. We denote the *traded payoffs* (also reffered to as financial risks) by the vector $\boldsymbol{Y} = (Y_0, Y_1, Y_2, \ldots, Y_{n^f})$. The first asset, Y_0, denotes the payoff of a risk-free bank account earning the deterministic rate r, i.e. $Y_0 = \mathrm{e}^{rT}$. In this paper, we assume for simplicity that $r = 0$[1]. Examples of traded assets are stocks, bonds, options, longevity derivatives, etc. The unique time-0 price y_i of the payoff Y_i can be observed in the market. We assume that $y_0 = 1$.

The random vector $\boldsymbol{Y}$ is defined on the probability space $\left(\Omega, \mathcal{F}^f, \mathbb{P}^f\right)$, which we will refer to as the financial probability space. The set $\mathcal{F}^f$ denotes the σ-algebra generated by the financial risks, i.e. $\mathcal{F}^f = \sigma\left(\boldsymbol{Y}\right)$. We assume there is a risk neutral probability measure[2] $\mathbb{Q}^f$ which is equivalent to the real-world financial probability measure $\mathbb{P}^f$. The time-0 prices can be expressed as discounted expectations under this risk neutral measure:

$$y_i = \mathrm{e}^{-rT}\mathbb{E}_{\mathbb{Q}^f}\left[Y_i\right], \text{ for } i = 0, 1, 2, \ldots, n^f. \tag{2.1}$$

A *financial claim* S^f is an $\mathcal{F}^f$-measurable random variable. We denote the set of all financial claims by $\mathcal{C}^f$. A financial claim is a function of the financial risks:

$$S^f = h\left(\boldsymbol{Y}\right),$$

for some function[3] h.

The financial claim S^f is said to be *headgeable* if we can replicate the claim by using an appropriate linear combination of the available traded assets. Below we define the set of the hedgeable claims.

[1]Note that our results can be generalized to the situation where interest rates are deterministic.

[2]In our setting, the existence of the risk neutral probability measure is equivalent to the absence of arbitrage; see e.g. Dalang et al. (1990) and Delbaen and Schachermayer (2006).

[3]throughout the paper we assume that all functions we encounter are Borel measurable.

Definition 2.3 (Hedgeable claim) *A claim $S^h \in \mathcal{C}$ is a hedgeable claim if we have a real vector $\boldsymbol{\nu} \in \mathbb{R}^{n^f+1}$ such that*

$$S^h = \boldsymbol{\nu} \cdot \boldsymbol{Y},$$

almost surely. The set of hedgeable claims is denoted by $\mathcal{C}^h$. The vector $\boldsymbol{\nu}$ is called the hedge *of the claim S^h.*

We assume that the $n^f + 1$ tradeable assets are non-redundant, i.e. we cannot use a subset of the traded assets to replicate the payoff of another traded asset. Assume $\boldsymbol{\theta} \in \mathbb{R}^{n^f+1}$. Then $\boldsymbol{\theta} \cdot \boldsymbol{Y} = 0$ implies $\boldsymbol{\theta} = (0, 0, \ldots, 0)$. Since we assume the traded assets to be non-redundant, the hedge $\boldsymbol{\nu}$ is unique. Of course, a hedgeable claim is a financial claim, i.e. we have $\mathcal{C}^h \subseteq \mathcal{C}^f$. In case the market is complete, both sets coincide.

A hedgeable claim can be managed by buying at time $t = 0$ the hedge $\boldsymbol{\nu}$ at the market price $\boldsymbol{\nu} \cdot \boldsymbol{y}$. Since the hedge will replicate the payoff of the liability, the residual liability, i.e. the liability after taking into account the payout of the hedge, is zero. Therefore, in order to avoid arbitrage opportunities, the price of a headgeable claim should be equal to the unique price of the hedge. It then follows from (2.1) that the price of a headgeable claim should be determined using a risk neutral expectation. The financial market, however, is assumed to be incomplete, which implies that there are financial claims which cannot be perfectly replicated.

2.3 Actuarial claims

There are n^a *actuarial risks* denoted by $X_1, X_2, \ldots, X_{n^a}$. We also use the notation $\boldsymbol{X}$ to denote the vector with all actuarial risks. The actuarial risks are defined on the actuarial probability space $(\Omega, \mathcal{F}^a, \mathbb{P}^a)$, where $\mathcal{F}^a$ is the σ-algebra generated by the actuarial risks $\boldsymbol{X}$, i.e. $\mathcal{F}^a = \sigma(\boldsymbol{X})$. An *actuarial claim* is defined as an $\mathcal{F}^a$-measurable random variable and the set of all actuarial claims is denoted by $\mathcal{C}^a$. An example of an actuarial risk is the future lifetime of a policyholder.

The actuarial risks are not traded on a public exchange and therefore one cannot use the hedging approach to manage an actuarial claim or risk neutral valuation to determine the price of an actuarial claim. However, the law of large numbers states that by aggregating independent actuarial risks, the corresponding actuarial claim will converge to its expected value. Therefore, the valuation of actuarial risks should be based on the expected loss, under the real-world probability measure, augmented by a safety loading to account for the imperfect diversification. Note, however, that we do not assume the actuarial risks to be independent. We allow, for example, for the situation where the actuarial risks are only conditionally independent.

A valuation ρ^a is said to be an actuarial valuation if it can be written in the following form:

$$\rho^a[S] = \mathbb{E}[S] + \text{RM}[S],$$

where RM$[S]$ denotes a risk margin for the claim S. Note that the risk margin and the expectation are both determined using the real-world probability measure.

The risk margin can be determined by using the standard deviation, resulting in the standard deviation principle:

$$\text{Standard Deviation Principle:} \quad \rho^a[S] = \mathbb{E}[S] + \beta\sqrt{\text{Var}[S]}, \tag{2.2}$$

where $\beta \in \mathbb{R}$ is a safety loading. Throughout the paper, we will assume that the actuarial valuation principle is the standard deviation principle. However, our results can be applied for other types of actuarial valuations. An overview of different approaches for actuarial pricing can be found in Borch (1974), Bühlmann (1980), Reich (1986), Laeven and Goovaerts (2008) and Kaas et al. (2008).

2.4 Systematic claims

Systematic risks are not traded on a public exchange. Hence, they cannot be managed using the hedging approach. Moreover, diversification is not a feasible risk management strategy when facing systematic risks. Diversifying a portfolio consists of aggregating a large number of independent and identical copies of the same risk. A systematic risk, e.g. longevity risk, has the same impact on different individual risks. Longevity risk, for example, impacts the future lifetime of a large group of policyholders simultaneously in the same direction. Therefore, adding more policyholders in your portfolio from this same group, will not lead to diversification effects. Although the longevity risk of different groups may be independent (e.g. longevity in the United States and China), one cannot find a large number of independent copies. Other examples of systematic risks are inflation risk, interest rate risk and catastrophe risk. Non-traded financial assets (i.e. non-traded stocks) are also members of the class of systematic risks.

We denote these systematic risks by the random vector $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{n^s})$. The systematic risks $\boldsymbol{Z}$ can be dependent on the traded risks $\boldsymbol{Y}$ and the actuarial risks $\boldsymbol{X}$. Denote by$\mathcal{F}^s$ the $\sigma-$algebra generated by the systematic risks $\boldsymbol{Z}$, i.e. $\mathcal{F}^s = \sigma(\boldsymbol{Z})$. The systematic probability space is denoted by $(\Omega, \mathcal{F}^s, \mathbb{P}^s)$. A *systematic claim* is an $\mathcal{F}^s$-measurable random variable and the set of all systematic claims is denoted by $\mathcal{C}^s$.

Denote by $\mathcal{F}^{f,s}$ the $\sigma-$algebra generated by the financial and the systematic risks, i.e.

$$\mathcal{F}^{f,s} = \mathcal{F}^f \bigvee \mathcal{F}^s. \tag{2.3}$$

Claims containing only financial and systematic risks are in the set $\mathcal{C}^{f,s}$. If $S^{f,s} \in \mathcal{C}^{f,s}$, then $S^{f,s}$ is an $\mathcal{F}^{f,s}$-measurable random variable. The claim $S^{f,s}$ is not completely hedgeable since it also depends on systematic risks. However, the claim is not driven by the actuarial risks and therefore cannot be managed by using diversification.

The systematic part of a claim should be priced with an appropriate systematic valuation principle. A valuation principle for financial and actuarial claims can be derived from an underlying risk management strategy. However, since systematic risks are neither hedgeable nor diversifiable, it is not straightforward to determine a class of systematic valuations. In this section, we will discuss a special class of valuations which can be employed to derive the value of the systematic part of a claim: linear systematic valuations. Note, however, that the methodology we introduce in this paper to value hybrid claims is not limited to our particular choice of the systematic valuation.

Assume the systematic valuation ρ^s is a linear valuation. Moreover, we assume ρ^s can be expressed as follows:

$$\rho^s[S] = \mathbb{E}[\varphi \times S], \tag{2.4}$$

for some positive random variable φ. We assume the random variable φ is $\mathcal{F}^{f,s}$-measurable with $\mathbb{E}[\varphi] = 1$. The systematic valuation ρ^s uses the expectation of the claim, but only after transforming the claim S with an appropriate 'distortion' φ. The distortion allows to use a prudent valuation approach by considering the claim in a less preferable, or stressed, scenario. The value $\rho^s[S]$ can then be interpreted as the expected payout of the claim S in the stressed scenario.

In order to determine the distortion φ, we follow the approach of Deelstra et al. (2020) and employ the Esscher transform, which was introduced in Esscher (1932) and applied to price derivatives in Gerber and Shiu (1995).

$$\text{Esscher distortion: } \varphi = \frac{\mathrm{e}^{-\sum_{i=1}^{n^s} w_i Z_i - \sum_{i=1}^{n^f} v_i Y_i}}{\mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{n^s} w_i Z_i - \sum_{i=1}^{n^f} w_i Y_i}\right]}, \tag{2.5}$$

for some constants $w_1, w_2, \ldots, w_{n^s}$ and $v_1, v_2, \ldots, v_{n^f}$. These constants have to be determined by using an appropriate set of conditions on the valuations of financial and systematic claims. For example, one may require the systematic valuation to be consistent with the available market prices. This results in the following conditions:

$$\mathbb{E}[\varphi Y_i] = y_i, \text{ for } i = 1, 2, \ldots, n^f. \tag{2.6}$$

In order to obtain an additional n^s conditions such that we can calibrate the Esscher transform in (2.5), we can require a given risk margin γ_i for the systematic risk Z_i. For example, in Zeddouk and Devolder (2019), the authors argue that a suitable systematic valuation should result in a risk margin which is consistent with the Solvency II risk margin using the Cost-of-Capital approach. This leads to an additional set of conditions:

$$\mathbb{E}[\varphi Z_i] = \mathbb{E}[Z_i] + \gamma_i, \text{ for } i = 1, 2, \ldots, n^s. \tag{2.7}$$

Combining (2.6) and (2.7) provides a set of conditions for determining the Esscher transformation (2.5).

Example 1 (Systematic valuation if financial and systematic risks are independent) In order to determine the systematic valuation (2.4) with Esscher transform (2.5), we need a joint calibration of the weights $w_1, w_2, \ldots, w_{n^s}$ and $v_1, v_2, \ldots, v_{n^f}$. In order to simplify the calculations, assume for this example that financial risks are independent from the actuarial and systematic risks. Moreover, we assume that the systematic valuation ρ^s can be expressed as

$$\rho^s[S] = \mathbb{E}\left[\left(\varphi^f \times \varphi^s\right) S\right], \tag{2.8}$$

where $\mathbb{E}\left[\varphi^f\right] = 1$, φ^f is $\mathcal{F}^f$-measurable and φ^s is $\mathcal{F}^s$-measurable. Then it follows that $\mathbb{E}[\varphi^s] = 1$.

From (2.8), we find:

$$\rho^s\left[S^f\right] = \mathbb{E}\left[\varphi^f S^f\right], \text{ for } S^f \in \mathcal{C}^f. \tag{2.9}$$

Expression (2.9) states that φ^f is the financial distortion. One may require the systematic valuation to be consistent with a given risk neutral probability measure. If we take $\varphi^f = \frac{\mathrm{d}\mathbb{Q}^f}{\mathrm{d}\mathbb{P}^f}$, then we find that

$$\rho^s\left[S^f\right] = \mathbb{E}_{\mathbb{Q}^f}\left[S^f\right], \text{ for } S^f \in \mathcal{C}^f. \tag{2.10}$$

Expression (2.10) shows that by tuning the distortion φ^f in an appropriate way, the systematic valuation ρ^s is using risk neutral valuation for all financial claims. The risk neutral measure $\mathbb{Q}$ may be estimated from available market information. However, there are only finitely many traded assets and infinitely many possible risk neutral measures. The choice of the risk neutral measure is not straightforward. One can use the minimal entropy martingale measure (Frittelli (2000) and Dhaene et al. (2015)), the Esscher transform (Gerber and Shiu (1995), Hubalek and Sgarra (2006) and Goovaerts and Laeven (2008)) or the stochastic discount factor approach (see Hansen and Jagannathan (1997)).

For systematic claims $S^s \in \mathcal{C}^s$, the independence assumption leads to

$$\rho^s\left[S^s\right] = \mathbb{E}\left[\varphi^s S^s\right], \text{ for } S^s \in \mathcal{C}^s. \tag{2.11}$$

Expression (2.11) shows that the distortion φ^s can be calibrated solely using the systematic claims. We conclude that in this special situation where systematic and financial risks are independent, we can determine the systematic valuation defined in (2.8) in two steps. The traded assets will result in an estimate for φ^f whereas one can impose conditions on the risk margin of the systematic risks to calibrate the distortion φ^s.

2.5 Hybrid claims

The financial, actuarial and systematic probability space are subspaces of the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{F} = \mathcal{F}^f \bigvee \mathcal{F}^s \bigvee \mathcal{F}^a$. All $\mathcal{F}$-measurable random variables are in the set $\mathcal{C}$. The probability measure $\mathbb{P}$ is such that it is consistent with the financial, systematic and actuarial probability measures. A claim $S \in \mathcal{C}$ is a time-T payoff which is a combination of actuarial, systematic and traded risks:

$$S = f\left(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\right),$$

for some function $f : \mathbb{R}^n \to \mathbb{R}$, where $n = 1 + n^f + n^s + n^a$.

In case a claim is independent from the financial assets, the financial prices do not contain any information about the value of the hybrid claim. We denote by $\mathcal{C}^\perp$ the set of all claims which are independent of the financial risks $\boldsymbol{Y}$:

$$S^\perp \in \mathcal{C}^\perp \Leftrightarrow S^\perp \perp \boldsymbol{Y}.$$

The claims in the set $\mathcal{C}^\perp$ are called *orthogonal claims*; see also Dhaene et al. (2017). The real-world distribution of an orthogonal claim should be employed to build a model to determine the value of such a claim.

The set $\mathcal{C}^{\perp,a}$ denotes the set of the orthogonal actuarial risks:

$$\mathcal{C}^{\perp,a} = \left\{S \in \mathcal{C}^a \middle| \, S \text{ is independent from } \boldsymbol{Y} \text{ and } \boldsymbol{Z}\right\}. \tag{2.12}$$

These risks are independent from the financial and systematic risk factors. A claim $S^a \in \mathcal{C}^{\perp,a}$ is also called a *pure actuarial claim*. Pure actuarial claims are assumed to be diversifiable and therefore, the valuation is based on the expectation, adjusted by a safety loading to account for fluctuations around the mean.

Define the set $\mathcal{C}^{\perp,s}$ containing the orthogonal systematic claims as follows

$$\mathcal{C}^{\perp,s} = \mathcal{C}^{\perp} \cap \mathcal{C}^{s}. \tag{2.13}$$

This set contains the systematic claims which are independent from the financial risks. The independence from the financial market implies that an orthogonal systematic claim cannot be hedged using the traded assets. However, the systematic nature of the claim makes the traditional actuarial principle inappropriate for such a claim. Therefore, we should use a given systematic valuation ρ^s for the valuation of orthogonal systematic claims.

Definition 2.4 *We say that S is a hybrid claim if it is not hedgeable, not pure actuarial or orthogonal systematic:*

$$S \in \mathcal{C} \setminus \left(\mathcal{C}^{h} \cup \mathcal{C}^{\perp,a} \cup \mathcal{C}^{\perp,s}\right).$$

The valuation of a hybrid claim will require a mixture of the risk neutral expectation, the traditional actuarial valuation and the systematic valuation.

3 Market-consistent and model-consistent valuations

3.1 Market- and model-consistency

A hybrid claim is a combination of the actuarial, systematic and financial risks. Therefore, a hybrid claim depends on the information about the traded assets $\boldsymbol{Y}$. It is then reasonable to impose that a valuation of a hybrid claim takes into account the market prices $\boldsymbol{y}$ of the traded assets. The valuation of the hybrid claim should be 'consistent' with the available market information. This observation leads to the market-consistent valuations.

Definition 3.1 (Market-consistency) *A valuation $\rho : \mathcal{C} \to \mathbb{R}$ is said to be a* market-consistent *valuation if for any claim $S \in \mathcal{C}$ and trading strategy $\boldsymbol{\nu}$, we have:*

$$\rho\left[S + \boldsymbol{\nu} \cdot \boldsymbol{Y}\right] = \rho[S] + \boldsymbol{\nu} \cdot \boldsymbol{y}. \tag{3.1}$$

If a hybrid claim can be decomposed in a hedgeable part $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ and a residual part S, then any market-consistent price is the sum of the prices of the hedgeable and residual part. One can buy the hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ at the price $\boldsymbol{\nu} \cdot \boldsymbol{y}$ and therefore the price of the headgeable part is unambiguously determined by its hedging cost. The residual part S, on the contrary, has to be priced using the valuation ρ. If the value of a hybrid claim is determined with a market-consistent valuation, this value is consistent with the prices of the available traded financial assets in that it takes into account the market price to buy the hedge.

If ρ is a market-consistent valuation, the value of a hedgeable claim $S^h \in \mathcal{C}^h$ solely depends on the available market information:

$$\rho\left[S^h\right] = \boldsymbol{\nu} \cdot \boldsymbol{y}, \tag{3.2}$$

where $\boldsymbol{\nu}$ is the hedge for S^h. We can use (2.1) to express the value of a hedgeable claim as the risk neutral expected value:

$$\rho\left[S^h\right] = \mathbb{E}_{\mathbb{Q}^f}\left[S^h\right]. \tag{3.3}$$

Expression (3.3) shows that the valuation of the hedgeable claim S^h does not require the real-world distribution of the claim S^h. Since the prices $\boldsymbol{y}$ can be observed in the market, it follows from Expression (3.2) that the value of a hedgeable claim can be determined in a model-free way.

A valuation is model free if the information about the support of the claim is sufficient to determine the value of the claim. As a result, the value of the claim is independent from the distribution function of the claim. If all claims can be replicated using available traded assets, a valuation which assigns the price of the unique replicating portfolio to a claim is a model-free valuation, provided the market prices of the traded assets can be observed. A valuation which is not model free, on the other hand, is a valuation that depends on the information about the distribution function of a claim. This distribution function is in most cases unknown. Expert judgement is therefore required to build an appropriate model to approximate the distribution function of the claim. Therefore, we refer to such valuations as model valuations and we denote the set of all model valuations by $\mathcal{M}$. The standard deviation principle defined in (2.2) is an example of a model valuation.

In the following definition, we define the set of valuations that take into account that employing the market information is not appropriate for an orthogonal claim. We call these valuations the *weak model-consistent valuations*.

Definition 3.2 (Weak model-consistency) *A valuation $\rho : \mathcal{C} \to \mathbb{R}$ is said to be a* weak model-consistent *valuation if there exists a model valuation $\pi \in \mathcal{M}$ such that for any orthogonal claim $S^\perp \in \mathcal{C}^\perp$, we have:*

$$\rho\left[S^\perp\right] = \pi\left[S^\perp\right]. \tag{3.4}$$

Definition 3.2 states that a weak model-consistent valuation is using a given model valuation π, different from ρ, for pricing an orthogonal claim. For general hybrid claims, the valuation ρ will be a combination of the hedging cost $\boldsymbol{\nu} \cdot \boldsymbol{y}$ and the model valuation π. The definition of weak model-consistency is equivalent with the definition of model-consistency which was proposed in Dhaene et al. (2017).

3.2 Hedge-based valuations

In this subsection we introduce the class of hedge-based valuations, which are valuations that are both market consistent and weak model consistent. In order to understand the hedge-based valuations, we first have to define the notion of a hedger. For a detailed discussion and more properties on hedgers and hedge-based valuations, we refer to Dhaene et al. (2017).

Definition 3.3 (Hedger) *A* hedger *is a function $\boldsymbol{\theta} : \mathcal{C} \to \mathbb{R}^{n^f+1}$ satisfying the following conditions*

- *Normalized:* $\boldsymbol{\theta}_0 = (0, 0, \ldots, 0)$.
- *Translation invariant:* $\boldsymbol{\theta}_{S+a} = \boldsymbol{\theta}_S + (a, 0, 0, \ldots, 0)$*, for* $S \in \mathcal{C}$ *and* $a \in \mathbb{R}$.

A hedger assigns an appropriate hedging strategy to a hybrid claim S. The hedging strategy $\boldsymbol{\theta}_S = \left(\theta_S^{(0)}, \theta_S^{(1)}, \ldots, \theta_S^{(n^f)}\right)$ is also called the hedge for the claim S. The component $\theta_S^{(i)}$ of the hedge determines the amount of units to be bought of the traded asset Y_i. It is reasonable to assume that the hedge of a hedgeable claim $\boldsymbol{\nu} \cdot \boldsymbol{Y}$ corresponds with $\boldsymbol{\nu}$. Consider the set $\mathcal{C}^{\perp}$ containing the claims that are independent of the traded assets $\boldsymbol{Y}$. It is also reasonable to assume that the hedge for a claim in $\mathcal{C}^{\perp}$ only consists of a position in the risk-free bank account.

Definition 3.4 (Fair hedgers) *Consider a hedge* $\boldsymbol{\nu}$ *and a claim* $S \in \mathcal{C}$*. A hedger* $\boldsymbol{\theta}$ *is:*

- *Market consistent if* $\boldsymbol{\theta}_{S+\boldsymbol{\nu}\cdot\boldsymbol{Y}} = \boldsymbol{\theta}_S + \boldsymbol{\nu}$.
- *Model consistent if* $\boldsymbol{\theta}_{S^{\perp}} = \left(\pi\left[S^{\perp}\right], 0, 0, \ldots, 0\right)$*, for a claim* $S^{\perp} \in \mathcal{C}^{\perp}$ *and some model valuation* $\pi \in \mathcal{M}$.
- *Fair if it is market and model consistent.*

The idea of market-consistent, model-consistent and fair hedgers was first introduced in Dhaene et al. (2017). The following theorem was proven in Theorem 3 of Dhaene et al. (2017) and characterizes the class of valuations that are both market and weak model consistent.

Theorem 3.1 *Consider a valuation* $\rho : \mathcal{C} \to \mathbb{R}$*. The following two statements are equivalent:*

1. *The valuation* ρ *is market consistent and weak model consistent.*
2. *The valuation* ρ *is a hedge-based valuation: there exist a model valuation* $\pi \in \mathcal{M}$ *and a fair hedger* $\boldsymbol{\theta}$ *such that*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right]. \tag{3.5}$$

The valuation defined in (3.5) is called a *hedge-based valuation* and was introduced in Dhaene et al. (2017) in a one-period setting. In Barigou and Dhaene (2019), Barigou et al. (2019) and Chen et al. (2021), the authors consider the hedge-based valuations in a multi-period setting whereas continuous-time versions of the hedge-based valuations are considered in Delong et al. (2019*a*,*b*).

A hedged-based valuation employs a hedger $\boldsymbol{\theta}$ to determine the hedgeable part of a hybrid claim. This hedgeable part can be priced using the available market information. The remaining, residual, part of the claim is then priced using an appropriate model valuation. We say that $S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}$ is the non-hedgeable part of the hybrid claim S.

3.3 2-step valuations

In this subsection we define an alternative class of valuations which are also market consistent and weak model consistent. For a detailed discussion of the 2-step valuation and its properties, we refer to Pelsser and Stadje (2014), where 2-step valuations are defined in a multi-period framework and a complete financial market. In this section, we follow the framework of Dhaene et al. (2017) and assume a single-period setting and an incomplete financial market.

We start by defining the notion of a conditional valuation; see Detlefsen and Scandolo (2005).

Definition 3.5 (Conditional valuation) *A* conditional valuation *is a function* $\pi\left[\cdot|\ \mathcal{F}^f\right]:\mathcal{C}\rightarrow\mathcal{C}^f$ *attaching a financial claim* $\pi\left[S|\ \mathcal{F}^f\right]\in\mathcal{C}^f$ *to a hybrid claim* $S\in\mathcal{C}$ *such that*

1. $\pi\left[\cdot|\ \mathcal{F}^f\right]$ *is normalized:* $\pi\left[0|\ \mathcal{F}^f\right]=0$*,*
2. $\pi\left[\cdot|\ \mathcal{F}^f\right]$ *is conditionally positive homogeneous:* $\pi\left[S\times S^f|\ \mathcal{F}^f\right]=S^f\times\pi\left[S|\ \mathcal{F}^f\right]$*, if* $S\geq 0$*.*
3. $\pi\left[\cdot|\ \mathcal{F}^f\right]$ *is conditionally translation invariant:* $\pi\left[S+S^f|\ \mathcal{F}^f\right]=\pi\left[S|\ \mathcal{F}^f\right]+S^f$*.*

The value of the financial part of a hybrid claim should be determined using a risk neutral probability measure, whereas a model valuation π should be used for the valuation of the non-financial part. Assume we know the realization of the financial risks. To be more precise, denote by $S_{\boldsymbol{Y}[\omega]}$ the random variable S conditional $\boldsymbol{Y}=\boldsymbol{Y}[\omega]$, for some $\omega\in\Omega$. For each ω, the claim $S_{\boldsymbol{Y}[\omega]}$ is a claim driven by actuarial and systematic risks. If we use the model valuation π to value this non-financial claim, we find the value $\pi\left[S_{\boldsymbol{Y}[\omega]}\right]$. The conditional risk measure $\pi\left[\cdot|\ \mathcal{F}^f\right]$ maps a claim S into a random variable such that

$$\pi\left[S|\ \mathcal{F}^f\right][\omega]=\pi\left[S_{\boldsymbol{Y}[\omega]}\right],\ \text{for each}\ \omega\in\Omega.$$

The 2-step valuation takes into account that the random variable $\pi\left[S|\ \mathcal{F}^f\right]$ corresponds with a financial claim that should be valuated using an appropriate risk neutral probability measure. In an incomplete market, the choice of the risk neutral distribution is not unique. Note, however, that each choice should be consistent with the traded asset prices.

Definition 3.6 (2-step valuation) *Consider a valuation* $\rho:\mathcal{C}\rightarrow\mathbb{R}$*. Then* ρ *is a* 2-step valuation *if we have that*

$$\rho[S]=\mathbb{E}_{\mathbb{Q}}\left[\pi\left[S|\ \mathcal{F}^f\right]\right],\tag{3.6}$$

where $\mathbb{Q}$ *is a risk neutral probability measure.*

The 2-step valuation was first introduced in Pelsser and Stadje (2014) and further investigated in Pelsser and Salahnejhad (2016), Salahnejhad and Pelsser (2020). It was shown in Dhaene et al. (2017) that any market- and weak model-consistent valuation is also a 2-step valuation. Therefore, we conclude that the class of hedge-based valuations is the same as the set of 2-step valuations, i.e. we can write:

$$\rho\ \text{is a 2-step valuation}\ \Leftrightarrow\rho\ \text{is a hedge-based valuation.}$$

It was shown in Dhaene et al. (2017) that this equivalence can be proven without assuming that the conditional valuation is conditional positive homogeneous.

The 2-step valuation first maps a hybrid claim into a financial claim by using an appropriate model valuation. In a second step, the remaining financial claim is valuated using a risk neutral expectation. Note that we do not assume the financial market to be complete. Therefore, a financial claim is not necessarily hedgeable and it may not be appropriate to use a risk neutral valuation for the financial claim. In Assa and Gospodinov (2018*a*) and Barigou and Linders (2021), the authors consider two-step valuations with non-linear financial valuation principles to take into account the valuation of financial claims in incomplete markets.

The 2-step valuation and the hedge-based valuations are combining market prices to price the hedgeable part of a hybrid claim, whereas the residual part is valuated using an appropriate model valuation.

4 Fair valuations with systematic risks

4.1 Definition of the 3-step hedge-based valuation

Market- and model-consistent valuations such as the hedge-based valuation defined in (3.5) and the two-step valuation defined in (3.6) use the risk neutral expectation to valuate the hedgeable part of a claim, whereas they use an appropriate actuarial valuation π to valuate the non-hedgeable part of the hybrid claim. By applying an actuarial valuation one assumes that the non-hedgeable part under consideration is diversifiable. This non-hedgeable part, however, is not always completely diversifiable and therefore a traditional actuarial valuation π may not always be appropriate.

Firstly, the financial market may be incomplete, implying that the financial part of the hybrid claim cannot be completely eliminated by an appropriate hedging strategy. As a result, the residual, non-hedgeable, part of the claim may contain financial risks that cannot be diversified by pooling together many contracts. Therefore, even in a world without systematic risks, the valuation of the residual part should not solely be based on a traditional actuarial valuation. If we use a traditional actuarial valuation π in the hedge-based valuation defined in (3.5), both the actuarial and the financial part of the residual claim are valuated using the same actuarial valuation. As a result, we do not have the flexibility to distinguish between the systematic nature of the financial risks and the diversifiable nature of the actuarial risks in the non-hedgeable part of the claim. The two-step valuation defined in (3.6), on the contrary, will use risk neutral valuation for both the hedgeable and the unhedgeable part of the financial claim and therefore does not take into account the incompleteness of the financial market.

Secondly, an orthogonal claim is not necessarily a diversifiable claim. Indeed, a claim that is independent from the financial market may depend on systematic, non-traded, risk factors. Therefore, the model valuation π should differentiate between the systematic part and the actuarial part of the claim. Both the hedge-based and the 2-step valuation are only differentiating between financial and non-financial risks. Therefore, systematic risks are valuated using the actuarial valuation.

The set $\mathcal{C}^{\perp,s}$ contains the systematic claims which are independent from the financial market (see (2.13)), i.e. it may not be appropriate to use risk neutral valuation for these claims. Moreover, since systematic claims are not diversifiable, traditional actuarial valuation principles may also not be appropriate. Therefore, we assume that a claim in the set $\mathcal{C}^{\perp,s}$ is valuated with a systematic valuation principle ρ^s, which may be different from the traditional actuarial valuation principle. Note that the systematic valuation may be based on the risk neutral distribution, see (2.6), whereas an actuarial valuation solely uses the real-world distribution.

Assume we have a weak model-consistent valuation ρ. Then ρ will use the model valuation π to price the residual claim after an appropriate hedge is taken into account. Since this residual part is a combination of systematic and actuarial risks, the model valuation π should be a combination of the systematic valuation ρ^s and the actuarial valuation ρ^a. We therefore introduce a subclass of valuations, which we call the strong model-consistent valuations, by requiring that systematic claims are valuated using the systematic valuation ρ^s and orthogonal actuarial claims are valuated using the actuarial valuation ρ^a.

Definition 4.1 (Strong model-consistency) *A valuation $\rho : \mathcal{C} \to \mathbb{R}$ is said to be* strong model consistent *if it is weak model consistent where the model valuation $\pi \in \mathcal{M}$ satisfies the following conditions:*

1. *Systematic valuation: For any claim $S^s \in \mathcal{C}^{f,s}$, we have:*

$$\pi\left[S^{f,s}\right] = \rho^s\left[S^{f,s}\right], \text{ if } S^s \in \mathcal{C}^{f,s}, \tag{4.1}$$

   where ρ^s is a systematic valuation.

2. *Actuarial valuation: For any orthogonal actuarial claim S^a, we have:*

$$\pi\left[S^a\right] = \rho^a\left[S^a\right], \text{ if } S^a \in \mathcal{C}^{\perp,a}, \tag{4.2}$$

   where ρ^a is an actuarial valuation.

Condition (4.2) states that only the pure actuarial claims, i.e. the claims that are independent of both the financial and systematic information should be priced with the actuarial valuation principle. Condition (4.1) states that the model valuation principle should use only the valuation ρ^s for claims that do not contain actuarial risks. Note however, that claims in $\mathcal{C}^{f,s}$ can be dependent on the actuarial risks.

Definition 4.2 (Fair valuation) *Consider a hybrid claim $S \in \mathcal{C}$ and a valuation ρ. We say that ρ is a* fair valuation *if it is market consistent and strong model consistent.*

Market-consistency of a fair valuation implies that a hedgeable claim should be valuated at its hedging cost. The strong model-consistency of a fair valuation implies that orthogonal claims should be valuated using a model valuation π. However, this model valuation π should distinguish between systematic and actuarial risks, since they require a different type of valuation. The actuarial valuation principle ρ^a should only be used for pure actuarial claims. The notion of a fair valuation was also defined in Dhaene et al. (2017), where the authors use a weaker definition. Indeed, they define a fair valuation as a valuation which is market consistent and weak model consistent.

In Definition 3.5 the $\mathcal{F}^f-$conditional valuation was introduced. Using this definition, but replacing $\mathcal{F}^f$ by $\mathcal{F}^{f,s}$ results in the class of $\mathcal{F}^{f,s}-$conditional valuations, which are valuations that map a hybrid claim into the set $\mathcal{C}^{f,s}$.

Assume we have an actuarial valuation ρ^a. Then we say that the $\mathcal{F}^{f,s}-$conditional valuation $\rho^a\left[\cdot|\ \mathcal{F}^{f,s}\right]:\mathcal{C}\to\mathcal{C}^{f,s}$ is a *conditional actuarial valuation* if

$$\rho^a\left[S^{\perp,a}|\ \mathcal{F}^{f,s}\right]=\rho^a\left[S^{\perp,a}\right],\ \text{for } S^{\perp,a}\in\mathcal{C}^{\perp,a}. \tag{4.3}$$

We can now define the *3-step hedge-based valuation*. We combine the hedging approach from the hedge-based market-consistent valuation and the conditional valuation approach from the 2-step valuation. In a first step, an appropriate hedging strategy $\boldsymbol{\theta}_S$ is used to offset the hedgeable part of a hybrid claim S. What remains is a residual claim $S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}$ which is mainly driven by the actuarial and systematic risks. The valuation of the residual claim should use the actuarial valuation for the actuarial part of the claim and the systematic valuation for the systematic part of the claim. Therefore, in the second valuation step, the residual claim is transformed in a systematic claim by using an $\mathcal{F}^{f,s}$-conditional valuation. Indeed, conditioning on the systematic and financial risk factors will transform the residual claim in a diversifiable claim for which an actuarial valuation principle can be used. The last, and third step consists of valuating the remaining claim using a systematic valuation principle.

Definition 4.3 (3-step hedge-based valuation) *A valuation ρ is said to be a* 3-step hedge-based valuation *if for any claim $S\in\mathcal{C}$, it can be expressed as follows*

$$\rho[S]=\boldsymbol{\theta}_S\cdot\boldsymbol{y}+\rho^s\left[\rho^a\left[S-\boldsymbol{\theta}_S\cdot\boldsymbol{Y}|\ \mathcal{F}^{f,s}\right]\right], \tag{4.4}$$

where $\boldsymbol{\theta}$ is a fair hedger, ρ^s is a systematic valuation principle and ρ^a is a conditional actuarial valuation principle.

4.2 Properties

The following lemma shows that the 3-step hedge-based valuation is a special case of the hedge-based valuations.

Lemma 4.1 *If the valuation ρ is a 3-step hedge-based valuation, then*

1. *ρ is also a hedge-based valuation.*
2. *ρ is also a 2-step valuation.*

Proof. We first prove **1**. Consider the 3-step valuation ρ given by (4.4). Define the valuation π as follows: $\pi[S] = \rho^s\left[\rho^a\left[S|\,\mathcal{F}^{f,s}\right]\right]$, then π is a model-valuation and therefore we directly find that ρ is a hedge based valuation.

In order to prove statement **2**, we use Theorem 8 of Dhaene et al. (2017), which shows that a hedge-based valuation is also a 2-step valuation. ■

Lemma 4.1 shows that the 3-step hedge-based valuations constitute a subset of the hedge-based valuations. Indeed, the 3-step valuations arise by using a particular choice of the model valuation that is used for the residual, non-hedgeable, part of the claim. The 3-step hedge-based valuations are also a subset of the 2-step valuations.

Theorem 4.1 *The 3-step hedge-based valuation ρ is a fair valuation.*

Proof. Assume ρ is given by (4.4). Since ρ is also a hedge-based valuation, we find from Theorem 3 in Dhaene et al. (2017) that ρ is market consistent and weak model consistent. Moreover, from Lemma 4.1, it follows that the model valuation π is given by $\pi[S] = \rho^s\left[\rho^a\left[S|\,\mathcal{F}^{f,s}\right]\right]$. It is then straightforward to verify that Conditions (4.1) and (4.2) hold for this choice of π. Therefore, ρ is strong market consistent, which proves that ρ is a fair valuation. ■

In the following proposition, we show that the value of a systematic claim which is independent of the financial risks is determined by the systematic valuation ρ^s. A pure actuarial claim is valued by using only the actuarial valuation.

Proposition 1 *Consider a 3-step hedge-based valuation ρ. Then:*

1. *Orthogonal systematic claim:*
$$\rho[S^s] = \rho^s[S^s], \textit{ for } S^s \in \mathcal{C}^{\perp,s}.$$

2. *Pure actuarial claim:*
$$\rho[S^a] = \rho^a[S^a], \textit{ for } S^a \in \mathcal{C}^{\perp,a}.$$

Proof. Consider an orthogonal systematic claim $S^s \in \mathcal{C}^{\perp,s}$. This implies that $S^s \in \mathcal{C}^{\perp}$. Since $\boldsymbol{\theta}$ is a fair hedger (and therefore also an actuarial hedger), we find that there is a model valuation π such that $\boldsymbol{\theta}_{S^s} = (\pi[S^s], 0, \ldots, 0)$. Then we find

$$\begin{aligned} \rho[S^s] &= \pi[S^s] + \rho^s\left[\rho^a\left[S^s - \boldsymbol{\theta}_{S^s}\cdot \boldsymbol{Y}|\,\mathcal{F}^{f,s}\right]\right] \\ &= \rho^s\left[\rho^a\left[S^s|\,\mathcal{F}^{f,s}\right]\right]. \end{aligned}$$

Since S^s is $\mathcal{F}^{f,s}-$measurable, we find $\rho[S^s] = \rho^s[S^s]$.

Consider an orthogonal actuarial claim $S^a \in \mathcal{C}^{\perp,a}$. We find $\boldsymbol{\theta}_{S^a} = (\pi[S^a], 0, \ldots, 0)$, and therefore

$$\rho[S^a] = \rho^s\left[\rho^a\left[S^a|\,\mathcal{F}^{f,s}\right]\right].$$

The orthogonal actuarial claim S^a is independent from both the financial and the systematic risks. Since ρ^a is a conditional actuarial valuation, we find $\rho^a\left[S^a|\,\mathcal{F}^{f,s}\right] = \rho^a[S^a] \in \mathbb{R}$. We can then conclude that $\rho[S^a] = \rho^a[S^a]$. ■

Theorem 4.2 *Consider a fair valuation ρ with a linear systematic valuation ρ^s. Then, there exists a conditional actuarial valuation such that*

$$\rho[S] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^s \left[\rho^a \left[S - \boldsymbol{\theta} \cdot \boldsymbol{Y} \middle|\, \mathcal{F}^{f,s}\right]\right].$$

Proof. If ρ is a fair valuation, it is also a market-consistent and weak model-consistent valuation. It then follows from Theorem 3.1 that there is a model valuation π such that

$$\rho\left[S\right] = \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \pi\left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y}\right],$$

where $\boldsymbol{\theta}$ is a fair hedger. Assume ρ^s is a linear valuation given by (2.4). Define the hedger $\boldsymbol{\nu}$ as follows:

$$\boldsymbol{\nu}_S = (\pi[S], 0, \ldots, 0).$$

Since ρ^s is linear, it is homogeneous and therefore we find:

$$\pi[S] = \rho^s\left[\boldsymbol{\nu}_S \cdot \boldsymbol{Y}\right]. \tag{4.5}$$

Note that the conditional valuation $\boldsymbol{\nu}_S \cdot \boldsymbol{Y}$ is $\mathcal{F}^{f,s}$-measurable and therefore we can define the conditional actuarial valuation as follows:

$$\rho^a\left[S \middle|\, \mathcal{F}^{f,s}\right] = \boldsymbol{\nu}_S \cdot \boldsymbol{Y}. \tag{4.6}$$

Combining (4.5) and (4.6) proves the result. ∎

Theorem 4.2 shows that in case the systematic valuation is the linear valuation defined in (2.4), a fair valuation has to be a 3-step hedge-based valuation. In case the systematic valuation is not linear, but coherent, a similar equivalence can be proven but then one needs to require the valuation ρ to be coherent as well; see Pelsser and Stadje (2014), Assa and Gospodinov (2018*b*) and Barigou and Linders (2021) .

4.3 Example: Product claim

Consider a financial market where a zero coupon bond is traded with maturity $T = 1$. Today's value of the bond is $y_0 = 1$ and the future time-1 value is $Y_0 = 1$. There is also a stock traded at a spot price y_1. The time-1 price of the stock is denoted by Y_1. The stock can go up to 100 or decrease to 50, i.e. $Y_1 \in \{50, 100\}$. The inflation during the next year is random and modelled using the positive random variable Z. We have a policyholder who will receive the amount Y_1, adjusted for inflation, at time $T = 1$, provided this policyholder is still alive at that time. The random variable X is an indicator random value which is equal to 1 if the policyholder survives and zero otherwise. The claim S is given by

$$S = Y_1 \times Z \times X. \tag{4.7}$$

For simplicity, we assume here that the future stock price Y_1 is independent from the actuarial and systematic risks.

In this example, we have two traded assets, $\boldsymbol{Y} = (Y_0, Y_1)$ and the claim S depends on the evolution of the asset prices. One can therefore search for a hedging strategy which partially offsets the liability S. The random variable X is an actuarial risk. Indeed, by selling a large amount of identical contracts to policyholders with independent and identical distributed future lifetimes, the average number of survivors will be close to the expectation $\mathbb{E}[X]$. The inflation Z is a systematic risk factor affecting the payout of different identical contracts in the same way. However, unlike the stock, the inflation is not traded and therefore one cannot hedge the inflation exposure using an adequate hedge.

The actuarial and systematic risks can be dependent. We assume that the marginal probabilities under the real-world probability measure are as follows:

$$\mathbb{P}[X = 1] = \mathbb{P}[X = 0] = \frac{1}{2}, \tag{4.8}$$

$$\mathbb{P}[Z = z_1] = \mathbb{P}[Z = z_2] = \frac{1}{2}. \tag{4.9}$$

One can use different probabilities than the one stated in assumptions (4.8) and (4.9). However, this particular choice will result in a deterministic conditional variance, which results in more transparent formulas. Moreover, we will also briefly consider the more general situation.

4.3.1 The hedge for S

We assume that the hedge of a claim S is determined by the mean-variance hedger $\boldsymbol{\theta}_S$. It is well known that the mean-variance hedge can be determined as follows (see also Dhaene et al. (2017)):

$$\text{Var}_{\mathbb{P}}[Y_1]\,\theta_S^{(1)} = \text{Cov}_{\mathbb{P}}[Y_1, S] \tag{4.10}$$

$$\theta_S^{(0)} = \mathbb{E}_{\mathbb{P}}[S] - \mathbb{E}_{\mathbb{P}}[Y_1]\,\theta_S^{(1)}. \tag{4.11}$$

We then find that $\theta_S^{(1)} = \mathbb{E}_{\mathbb{P}}[XZ]$. The number of risk-free bonds one has to buy is denoted by $\theta_S^{(0)}$ and in our particular case we can use the independence to show that $\theta_S^{(0)} = 0$. In order to acquire the hedging strategy, one has to invest in the appropriate amount of bonds and stocks. The price for buying the hedging strategy can then be expressed as follows:

$$\text{Price of the hedging strategy} = \theta_S^{(1)}\mathbb{E}_{\mathbb{Q}}[Y_1],$$

where $\mathbb{Q}$ is a risk neutral measure which is consistent with the financial pricing measure $\mathbb{Q}^f$ defined in (2.1).

4.3.2 The actuarial valuation principle

We use the standard deviation principle defined in (2.2) as the actuarial valuation principle. The conditional actuarial valuation principle is using the conditional expectation and the conditional variance:

$$\rho^{(a)}\left[S|\,\mathcal{F}^{f,s}\right] = \mathbb{E}_{\mathbb{P}}\left[S|\,\mathcal{F}^{f,s}\right] + \beta\sqrt{\text{Var}\left[S|\,\mathcal{F}^{f,s}\right]}.$$

The actuarial risk X is assumed to be independent of the financial risk Y_1. Therefore, we have that

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[X|\,\mathcal{F}^{f,s}\right] &= \mathbb{E}_{\mathbb{P}}\left[X|\,Z\right],\\ \text{Var}\left[X|\,\mathcal{F}^{f,s}\right] &= \text{Var}\left[X|\,Z\right].\end{aligned}$$

Indeed, in order to determine the conditional actuarial valuation of the actuarial risk X, we need the conditional distribution of X, given Z.

Assume we denote the following conditional probability by $\alpha \in [0,1]$:

$$\mathbb{P}\left[X=1|\,Z=z_1\right]=\alpha. \tag{4.12}$$

Then one can verify that:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[X|\,Z\right] &= I\left(2\alpha-1\right)+(1-\alpha), &(4.13)\\ \text{Var}_{\mathbb{P}}\left[X|\,Z\right] &= \alpha(1-\alpha), &(4.14)\end{aligned}$$

where I is an indicator random variable which is defined as follows

$$I=\begin{cases}1 & \text{if } Z=z_1,\\ 0 & \text{if } Z=z_2.\end{cases}$$

A proof of (4.13) and (4.14) can be found in Appendix A.2

4.3.3 The 3 step valuation

The 3-step valuation for the claim S given by (4.7) is determined as follows

$$\rho^{3-\text{step}}[S]=\mathbb{E}_{\mathbb{Q}}\left[Y_1\right]\mathbb{E}_{\mathbb{P}}\left[XZ\right]+\rho^s\left[Y_1\left(Z\left((1-\alpha)+\beta\sqrt{\alpha(1-\alpha)}+I(2\alpha-1)\right)-\mathbb{E}_{\mathbb{P}}\left[XZ\right]\right)\right]. \tag{4.15}$$

The steps of the proof can be found in the Appendix A.3. Expression (4.15) provides a formula for the 3-step valuation of S in terms of unconditional random variables. Indeed, given the systematic valuation ρ^s, the value $\rho^{3-\text{step}}[S]$ can be determined in analytical form or by simulation from the random variables Y_1 and Z.

Alternatively, one can use Expressions (4.13) and (4.14) to write (4.15) as follows in terms of conditional probabilities:

$$\rho^{3-\text{step}}[S]=\mathbb{E}_{\mathbb{Q}}\left[Y_1\right]\mathbb{E}_{\mathbb{P}}\left[XZ\right]+\rho^s\left[Y_1\left(\mathbb{E}_{\mathbb{P}}\left[XZ|\,Z\right]-\mathbb{E}_{\mathbb{P}}\left[XZ\right]\right)+Y_1Z\beta\sqrt{\text{Var}\left[X|Z\right]}\right]. \tag{4.16}$$

Note that this expression does not rely on the assumptions (4.8) and (4.9).

Expression (4.16) sheds light on the different parts of the value of the claim S. The first term represents the price of the hedging strategy for the claim S. The second term uses the systematic valuation to determine the value of the residual, unhedgeable, part of the claim. The unhedgeable part can be decomposed in a systematic and an actuarial part. The quantity $\mathbb{E}_{\mathbb{P}}\left[XZ|\,Z\right]$

represents the updated number of stocks one has to buy, once the realization of the systematic risk is known. If this amount deviates substantially from the unconditional estimate $\mathbb{E}_{\mathbb{P}}[XZ]$, the hedge $\boldsymbol{\theta}_S$ will not be as adequate as anticipated, resulting in extra, unancticipated, losses. Therefore, $Y_1(\mathbb{E}_{\mathbb{P}}[XZ|\,Z]-\mathbb{E}_{\mathbb{P}}[XZ])$ represents the hedging error due to the systematic risk. The fluctuations of the actuarial risk around the best estimate are modeled by $\beta\sqrt{\text{Var}[X|Z]}$. Therefore, $Y_1Z\beta\sqrt{\text{Var}[X|Z]}$ represents the hedging error coming from the actuarial risks.

4.3.4 The independent case

Assume for a moment that $\alpha=\frac{1}{2}$. It follows from (4.8) and (4.12) that this situation corresponds with the case where the actuarial and systematic risks are independent. One can then express the 3-step valuation as follows:

$$\rho^{3-\text{step}}[S]=\mathbb{E}_{\mathbb{Q}}[Y_1]\,\mathbb{E}_{\mathbb{P}}[XZ]+\rho^s\left[Y_1\mathbb{E}_{\mathbb{P}}[X](Z-\mathbb{E}_{\mathbb{P}}[Z])+Y_1Z\beta\sqrt{\text{Var}_{\mathbb{P}}[X]}\right]. \tag{4.17}$$

The realized inflation Z may be different from the anticipated inflation $\mathbb{E}_{\mathbb{P}}[Z]$. Such deviations may result in a less efficient hedge and the mismatch is modeled using the term $Y_1\mathbb{E}_{\mathbb{P}}[X](Z-\mathbb{E}_{\mathbb{P}}[Z])$. The fluctuations around the expectation $\mathbb{E}_{\mathbb{P}}[X]$ will make the hedge less efficient. This risk is measured using the variance $\text{Var}_{\mathbb{P}}[X]$.

Different choices for the systematic valuation ρ^s will result in different expressions for the 3-step valuation. Assume that ρ^s is the linear valuation principle defined in (2.8) where the financial distortion φ^f is determined by (2.10). Then $\rho^s[Y_1]=\mathbb{E}_{\mathbb{Q}}[Y_1]$. One then finds:

$$\rho^{3-\text{step}}[S]=\mathbb{E}_{\mathbb{Q}}[Y_1]\times\rho^s[Z]\times\rho^a[X]. \tag{4.18}$$

This result shows that in our simple example, the value of a product claim is the product of the values. This result remains to hold for general distributions. We summarize the result in the following proposition.

Proposition 2 *Assume the random vectors $\boldsymbol{Y}$, $\boldsymbol{Z}$ and $\boldsymbol{X}$ are independent of each other. Consider the product claim*

$$S=X\times Y_1\times Z.$$

Assume the systematic valuation ρ^s is the linear valuation defined in (2.8), satisfying (2.10) and the fair hedger is the mean-variance hedger. Then

$$\rho^{3\text{-}step}[S]=\mathbb{E}_{\mathbb{Q}}[Y_1]\times\rho^s[Z]\times\rho^a[X].$$

Proof. Consider the mean variance hedge $\boldsymbol{\theta}_S$ of the claim S. From (4.10) and (4.11), we find

$$\theta_S^{(1)}=\mathbb{E}[XZ],$$

and $\theta_S^{(0)}=0$. We can write the 3-step valuation as follows

$$\begin{aligned}\rho^{3-\text{step}}[S]&=\mathbb{E}[XZ]\,\mathbb{E}_{\mathbb{Q}}[Y_1]+\rho^s\left[\rho^a\left[Y_1(XZ-\mathbb{E}[XZ])\mid\mathcal{F}^{f,s}\right]\right]\\&=\mathbb{E}[XZ]\,\mathbb{E}_{\mathbb{Q}}[Y_1]+\rho^s\left[Y_1\left(\rho^a\left[XZ|\,\mathcal{F}^{f,s}\right]-\mathbb{E}[XZ]\right)\right].\end{aligned}$$

If we use the conditional positive homogeneity of the conditional valuation $\rho\left[\cdot|\,\mathcal{F}^{f,s}\right]$ and (4.3), we find $\rho^a\left[XZ|\,\mathcal{F}^{f,s}\right]=Z\rho^a\left[X\right]$. If ρ^s is the linear valuation defined in (2.8), we can write

$$\rho^{3-\text{step}}\left[S\right]=\mathbb{E}\left[XZ\right]\mathbb{E}_{\mathbb{Q}}\left[Y_1\right]+\mathbb{E}\left[\varphi^f\varphi^s Y_1\left(Z\rho^a\left[X\right]-\mathbb{E}\left[XZ\right]\right)\right].$$

If we then use the independence assumption and the fact that $\rho^s\left[Z\right]=\mathbb{E}\left[\varphi^s Z\right]$, we find the desired result. ■

5 The additive 3-step valuation

5.1 Additive valuations

We assume that all actuarial risks are independent, conditional on the random vectors $\boldsymbol{Y}$ and $\boldsymbol{Z}$:

$$\mathbb{P}\left[X_1\leq x_1,\ldots,X_{n^a}\leq x_{n^a}|\,\boldsymbol{Y}=\tilde{\boldsymbol{y}},\boldsymbol{Z}=\tilde{\boldsymbol{z}}\right]=\prod_{i=1}^{n^a}\mathbb{P}\left[X_i\leq x_i|\,\boldsymbol{Y}=\tilde{\boldsymbol{y}},\boldsymbol{Z}=\tilde{\boldsymbol{z}}\right],$$

where $\tilde{\boldsymbol{z}}\in\mathbb{R}^{n^s}$ and $\tilde{\boldsymbol{y}}\in\mathbb{R}^{n^f+1}$.

We assume a hedger for the hybrid claim $S\in\mathcal{C}$ is available and denoted by $\boldsymbol{\theta}_S$. We assume that $\boldsymbol{\theta}_S$ is a fair hedger (e.g. the mean-variance hedger). The claim H_S^h is defined as:

$$H_S^h=\boldsymbol{\theta}_S\cdot\boldsymbol{Y}. \tag{5.1}$$

The claim H_S^h is a hedgeable claim, in the sense that there exists a trading strategy consisting in positions in the traded assets $\boldsymbol{Y}$ that can replicate the payoff of H_S^h. We thus have that $H_S^h\in\mathcal{C}^h$. We can then write the claim S as follows:

$$S=H_S^h+\left(S-H_S^h\right).$$

The claim S can be decomposed in a hedgeable part and a residual part. The residual part $\left(S-H_S^h\right)$ is what remains of S after we apply the hedger $\boldsymbol{\theta}_S$. If the hedger is adequate, we expect that $\left(S-H_S^h\right)$ is to a large extent driven by the actuarial and the systematic risks.

The claim H_S^s is defined as follows:

$$H_S^s=\mathbb{E}\left[S-H_S^h|\,\mathcal{F}^{f,s}\right]. \tag{5.2}$$

The claim H_S^s only depends on the evolution of the traded assets and the systematic risks and we have that

$$H_S^s=g\left(\boldsymbol{Y},\boldsymbol{Z}\right),$$

for some function g. Note, however, that if the hedge H_S^h is adequate, then $S-H_S^h$ should not depend ‘too strongly’ on the traded risks $\boldsymbol{Y}$, meaning that H_S^s is mainly driven by the systematic risks $\boldsymbol{Z}$. Therefore, we also refer to H_S^s as the systematic part of the claim S. Roughly speaking, H_S^s is what remains in S after we first apply an appropriate hedge and then consider the average loss for difference scenarios of the financial and systematic risks.

We define the claim H_S^a as follows

$$H_S^a = \left(S - H_S^h - H_S^s\right). \tag{5.3}$$

The random variable H_S^a denotes the part of the claim S that remains after the hedgeable and the systematic parts are removed. We refer to H_S^a as the actuarial part of the claim S.

Combining (5.1), (5.2) and (5.3), we can decompose the hybrid claim S as follows:

$$S = H_S^h + H_S^s + H_S^a. \tag{5.4}$$

Expression (5.4) decomposes the hybrid claim S in three parts: a hedgeable part H_S^h, a systematic part H_S^s and an actuarial part H_S^a. This decomposition was also introduced in Dhaene (2020) for product claims and Deelstra et al. (2020) for the situation where financial and actuarial risks are independent.

The following lemma considers the hedgeable, systematic and actuarial part when dealing with special claims. The proof is trivial and is therefore not considered.

Lemma 5.1 *Consider a hybrid claim $S \in \mathcal{C}$ and a fair hedger $\boldsymbol{\theta}$ with model valuation $\pi[S] = \mathbb{E}[S]$. Then we have the following:*

1. *If $S^h \in \mathcal{C}^h$ and $S \in \mathcal{C}$, then*

$$\begin{aligned} H_{S+S^h}^h &= H_S^h + H_{S^h}^h && (5.5)\\ H_{S+S^h}^s &= H_S^s, && (5.6)\\ H_{S+S^h}^a &= H_S^a. && (5.7)\end{aligned}$$

2. *If $S \in \mathcal{C}^{f,s}$ then*

$$H_S^a = 0, \text{ a.s.} \tag{5.8}$$

3. *If $S \in \mathcal{C}^{\perp,a}$ then*

$$\begin{aligned} H_S^h &= \mathbb{E}[S] && (5.9)\\ H_S^s &= 0 &&\\ H_S^a &= S - \mathbb{E}[S]. && (5.10)\end{aligned}$$

Expression (5.5) shows that the hedgeable part of a claim is absorbed by the hedger and does not affect the systematic or actuarial part. Expression (5.8) shows that if the claim S does not depend on the actuarial risks, then the actuarial part of the claim is always zero. In case we have an orthogonal actuarial claim, it follows from (5.9) that the hedge invests only in the risk-free bank account to cover the best estimate. The remaining part is only containing independent actuarial risks and therefore there is no systematic part; see (5.10).

Example 2 (The actuarial claim H^a and diversification) Consider an insurance company holding an aggregate hybrid claim S_N, where N is the number of policyholders in the portfolio. We assume that, given the financial risks $\boldsymbol{Y}$ and the actuarial risks $\boldsymbol{Z}$, the portfolio is diversifiable, i.e.

$$\text{Var}\left[S_N \middle| \mathcal{F}^{f,s}\right] \to 0, \text{ if } N \to +\infty. \tag{5.11}$$

We decompose the claim S_N in a hedgeable, systematic and actuarial part:

$$\begin{aligned} H_N^h &= \boldsymbol{\theta}_{S_N} \cdot \boldsymbol{Y} \\ H_N^s &= \mathbb{E}\left[S_N - H_N^h \middle| \mathcal{F}^{f,s}\right] \\ H_N^a &= S_N - H_N^h - H_N^s. \end{aligned}$$

We can then write the variance of the actuarial claim as follows:

$$\begin{aligned} \text{Var}\left[H_N^a\right] &= \mathbb{E}\left[\text{Var}\left[S_N - H_N^h - H_N^s \mid \mathcal{F}^{f,s}\right]\right] + \text{Var}\left[\mathbb{E}\left[S_N - H_N^h - H_N^s \middle| \mathcal{F}^{f,s}\right]\right] \\ &= \mathbb{E}\left[\text{Var}\left[S_N \mid \mathcal{F}^{f,s}\right]\right], \end{aligned}$$

where we used that H_N^s and H_N^h in $\mathcal{C}^{f,s}$. Taking into account (5.11), we conclude:

$$\text{Var}\left[H_N^a\right] \to 0, \text{ if } N \to +\infty. \tag{5.12}$$

We conclude that H_N^a is indeed an actuarial claim in the sense that one can diversify the risk by increasing the number of policies. $\triangle$

In the following theorem, we derive an upper bound for the 3-step valuation in case the systematic valuation is subadditive.

Theorem 5.1 *Consider a hybrid claim S which can be written as follows:*

$$S = H^h + H^s + H^a,$$

where $H^h \in \mathcal{C}^h$ and $H^s \in \mathcal{C}^{f,s}$. Consider the 3-step valuation ρ with systematic valuation ρ^s and actuarial valuation ρ^a. Define the valuation ρ^U as follows:

$$\rho^U[S] = \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[\rho^a\left[H^a \middle| \mathcal{F}^{f,s}\right]\right] + \rho^s\left[H^s\right]. \tag{5.13}$$

If ρ^s is subadditive, then

$$\rho[S] \leq \rho^U[S].$$

If ρ^s is linear, then

$$\rho[S] = \rho^U[S].$$

Proof. Recall that the systematic part H_S^s belongs to $\mathcal{F}^{f,s}$. The valuation ρ is the 3-step valuation and therefore, we have that

$$\begin{aligned} \rho[S] &= \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[\rho^a\left[S - H_S^h - H_S^s + H_S^s \middle| \mathcal{F}^{f,s}\right]\right] \\ &= \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[\rho^a\left[H_S^a \middle| \mathcal{F}^{f,s}\right] + H_S^s\right], \end{aligned}$$

where we used (5.3) in the last step. In case the systematic risk measure ρ^s is subadditive, we find $\rho[S] \leq \rho^U[S]$. If ρ^s is linear, we directly find $\rho[S] = \rho^U[S]$.

$\blacksquare$

In order to value a hybrid claim S which can be decomposed in 3 different types of claims, one may wish to value the different claims separately using a different, appropriate valuation for each type of claim. This approach leads to an additive three step valuation. In case the actuarial part H_S^a is independent from the financial and systematic risks, (i.e. $H_S^a \in \mathcal{C}^{\perp,a}$), then we find that $\rho^a \left[H^a \middle| \mathcal{F}^{f,s} \right] \in \mathbb{R}$. Using the translation invariance of ρ^s then leads to the following expression for the upper bound ρ^U:

$$\rho^U[S] = \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^a \left[H^a \right] + \rho^s \left[H^s \right]. \tag{5.14}$$

This valuation decomposes a hybrid claim into a hedgeable, systematic and actuarial part and valuates each component separately. The value of the hybrid claim is the sum of the individual valuations. Such a valuation is called an additive valuation.

Definition 5.1 (Additive 3-step valuation) *Consider a systematic valuation ρ^s, an actuarial valuation ρ^a and a fair hedger $\boldsymbol{\theta}$. The additive 3-step valuation ρ^+ is defined as follows*

$$\rho^+[S] = \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^a \left[H_S^a \right] + \rho^s \left[H_S^s \right], \tag{5.15}$$

for any hybrid claim $S \in \mathcal{C}$, where H_S^h, H_S^s and H_S^a are given by (5.1), (5.2) and (5.3), respectively.

Theorem 5.2 *The additive valuation ρ^+ is a 3-step hedge-based valuation.*

Proof. We define the conditional valuation $\pi \left[\cdot \middle| \mathcal{F}^{f,s} \right]$ as follows:

$$\pi \left[S \middle| \mathcal{F}^{f,s} \right] = \mathbb{E} \left[S \middle| \mathcal{F}^{f,s} \right] + \rho^a \left[S - \mathbb{E} \left[S \middle| \mathcal{F}^{f,s} \right] \right].$$

Then we find

$$\begin{aligned} \pi \left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \middle| \mathcal{F}^{f,s} \right] &= \mathbb{E} \left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \middle| \mathcal{F}^{f,s} \right] + \rho^a \left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} - \mathbb{E} \left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \middle| \mathcal{F}^{f,s} \right] \right] \\ &= H_S^s + \rho^a \left[H_S^a \right], \end{aligned}$$

where we used (5.2) and (5.3). We can write $\rho^+[S]$ as follows:

$$\begin{aligned} \rho^+[S] &= \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^s \left[H_S^s \right] + \rho^a \left[H_S^a \right] \\ &= \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^s \left[H_S^s + \rho^a \left[H_S^a \right] \right] \\ &= \boldsymbol{\theta}_S \cdot \boldsymbol{y} + \rho^s \left[\pi \left[S - \boldsymbol{\theta}_S \cdot \boldsymbol{Y} \middle| \mathcal{F}^{f,s} \right] \right]. \end{aligned}$$

It then follows from Definition 4.3 that ρ^+ is a 3-step hedge-based valuation. $\blacksquare$

If we combine Theorem 5.2 with Theorem 4.1, we can conclude that the additive valuation ρ^+ is a fair valuation, i.e. market consistent and strong model consistent.

Proposition 3 *Consider a hybrid claim S which can be decomposed as in (5.4) and the valuations ρ^U and ρ^+.*

a. *If the actuarial valuation ρ^a can be written as $\rho^a[X] = \mathbb{E}[u(X)]$ and $\rho^s[X] \geq \mathbb{E}[X]$, then:*

$$\rho^+[S] \leq \rho^U[S].$$

b. *If the actuarial valuation ρ^a is the expectation, we have that*

$$\rho^U[S] = \rho^{3-step}[S] = \rho^+[S].$$

Proof. We start with proving **a**. We have that the conditional actuarial valuation can be written as follows: $\rho^a\left[H^a | \mathcal{F}^{f,s}\right] = \mathbb{E}\left[u\left(H^a\right) | \mathcal{F}^{f,s}\right]$. Then:

$$\begin{aligned}
\rho^{(s)}\left[\rho^a\left[H^a | \mathcal{F}^{f,s}\right]\right] &= \rho^{(s)}\left[\mathbb{E}\left[u\left(H^a\right) | \mathcal{F}^{f,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[u\left(H^a\right) | \mathcal{F}^{f,s}\right]\right] \\
&= \rho^a\left[H^a\right].
\end{aligned}$$

In order to prove **b**, we start with noting that $H^a = S - H^h - H^s$. If ρ^a is linear, we can write the following:

$$\begin{aligned}
\rho^a\left[H^a | \mathcal{F}^{f,s}\right] &= \mathbb{E}\left[S - H^h - H^s | \mathcal{F}^{f,s}\right] \\
&= \mathbb{E}\left[S - H^h | \mathcal{F}^{f,s}\right] - H^s.
\end{aligned}$$

Taking into account (5.2), we find $\rho^a\left[H^a | \mathcal{F}^{f,s}\right] = 0$, a.s. and therefore

$$\rho^U[S] = \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[H^s\right].$$

Similarly, we can determine ρ^{3-step} as follows

$$\begin{aligned}
\rho^{3-step}[S] &= \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[\rho^a\left[S - H^h | \mathcal{F}^{f,s}\right]\right] \\
&= \boldsymbol{\theta} \cdot \boldsymbol{y} + \rho^s\left[H^s\right],
\end{aligned}$$

where we used (5.2) and the linearity of ρ^a. For the additive 3-step valuation, we use that $\rho^a\left[H^a_S\right] = 0$, which proves the result.

∎

5.2 Example

In the following example we determine the 3-step hedge based valuation and the additive 3-step valuation for a product claim. Moreover, we assume that financial and actuarial risks are independent. This example was also considered in Dhaene (2020) for the hedge-based valuation and in Deelstra et al. (2020) for the additive 3-step valuation.

Consider an insurance company holding a portfolio with hybrid liabilities. The portfolio consists of a total of n^a policyholders. The payoff for policyholder i is given by:

$$\text{Payoff to policyholder } i = S^h \times g_i\left(X_i, \boldsymbol{Z}\right).$$

S^h is a headgeable claim, i.e. a linear combination of the traded stocks. The function g_i is a known function. The risk X_i is policyholder specific whereas $\boldsymbol{Z}$ contains non-traded risks which are affecting all policyholders (such as longevity risk). For this example, we assume that $\boldsymbol{Z}$ and $\boldsymbol{X}$ are independent of the financial risks $\boldsymbol{Y}$.

The per-policy liability to the insurance company is then given by:

$$S = S^h \times \frac{\sum_{i=1}^{n^a} g_i\left(X_i, \boldsymbol{Z}\right)}{n^a}. \tag{5.16}$$

Recall that the risks X_i are conditionally independent. We assume for simplicity that $g_i = g_1$, for all $i = 1, 2, \ldots, n^a$ and $X_1, X_2, \ldots, X_{n^a}$ are identical. However, all results can be generalized to allow for heterogeneous policyholders and payoff functions. Moreover, we write g_i instead of $g_i(X_i, \boldsymbol{Z})$ if no confusion is possible.

Assume the hedger $\boldsymbol{\theta}$ is the mean-variance hedger. One can then show that the hedgeable part of the claim is given by:

$$H^h = S^h \mathbb{E}\left[g_1\right].$$

Then, the systematic part H^s defined in (5.2) can be written as follows:

$$H^s \quad = \quad S^h \mathbb{E}\left[\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1\right]\right) \mid \mathcal{F}^s\right],$$

where we used that the functions g_i do not depend on the traded risks $\boldsymbol{Y}$. In order to ease the notation, we write $\mathbb{E}\left[\cdot\mid \boldsymbol{Z}\right]$ instead of $\mathbb{E}\left[\cdot\mid \mathcal{F}^s\right]$. We then find:

$$H^s = S^h\left(\mathbb{E}\left[g_1\mid \boldsymbol{Z}\right] - \mathbb{E}\left[g_1\right]\right).$$

For the actuarial part H^a given by (5.3), we find

$$H^a \quad = \quad S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1\mid \boldsymbol{Z}\right]\right).$$

We first derive the 3-step valuation ρ^{3-step}.

Proposition 4 *The 3-step valuation $\rho^{3-step}[S]$ for the product claim S given by (5.16) can be expressed as follows*

$$\rho^{3-step}[S] = \mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \rho^s\left[S^h\left(\mathbb{E}\left[g_1\mid \boldsymbol{Z}\right] - \mathbb{E}\left[g_1\right] + \beta\sqrt{\frac{Var\left[g_1\mid \boldsymbol{Z}\right]}{n^a}}\right)\right]. \tag{5.17}$$

A proof of this result can be found in Appendix A.4. The fist term in (5.17) corresponds with the price we have to pay if we buy the replicating portfolio for the hedgeable claim H_S^h. If the diversifiable part of the claim is perfectly diversified, the replicating portfolio will also replicate the hybrid claim S. However, there will be hedging errors caused by fluctuations of the non-hedgeable part around the estimate $\mathbb{E}\left[g_1\right]$. Indeed, the number of units of the hedgeable claim

S^h we are holding is $\mathbb{E}[g_1]$, wheareas the number of units we actually need to pay the liability is equal to $\frac{\sum_{i=1}^{n^a} g_i}{n_a}$. The second term in (5.17) considers the valuation of the hedging error. There are two sources that cause hedging errors: systematic and unsystematic deviations from the mean $\mathbb{E}[g_1]$. The systematic part is captured by the random variable $\mathbb{E}[g_1 | \boldsymbol{Z}] - \mathbb{E}[g_1]$ whereas the unsystematic fluctuations are captured by $\frac{\text{Var}[g_1 | \mathcal{F}^{f,s}]}{n^a}$.

We can now determine the additive valuation ρ^+ for this example.

Proposition 5 *The additive valuation $\rho^+[S]$ for the claim S given by (5.16) can be expressed as follows*

$$\rho^+[S] = \mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \rho^s\left[S^h\left(\mathbb{E}[g_1 | \boldsymbol{Z}] - \mathbb{E}[g_1]\right)\right] + \beta\sqrt{\mathbb{E}\left[\left(S^h\right)^2\right]}\sqrt{\mathbb{E}\left[\frac{Var[g_1 | \boldsymbol{Z}]}{n^a}\right]}. \quad (5.18)$$

A proof of this result can be found in Appendix A.5. The 3-step valuation (5.17) and the additive 3-step valuation (5.18) differ in the valuation of the unhedgeable part. The 3-step valuation aggregates the systematic and unsystematic fluctuations around the mean $\mathbb{E}[g_1]$ and can take into account dependencies between these two types of fluctuations. The additive valuation uses a separate valuation for the systematic and the unsystematic part, not taking into account the dependence between the two types of risks. Note that if we assume ρ^s to be the linear valuation defined in (2.8), we find back Expression (4.7) in Deelstra et al. (2020).

Note also that if ρ^s is a linear valuation, the 3-step valuation (5.17) will not reduce to the additive 3-step valuation (5.18). The reason is that the hedgeable claim S^h is determining the absolute size of the mismatch between the replicating portfolio and the hybrid claim. The 3-step valuation uses the systematic valuation to value the effect of the hedgeable claim. The additive 3-step valuation uses the systematic valuation when considering the effect of the hedgeable claim on the systematic fluctuations, whereas the actuarial valuation is used when considering the effect of the hedgeable claim on the unsystematic fluctuations.

In the next proposition, we determine the hedge-based valuation and the two step valuation. The proof can be found in Appendix A.6.

Proposition 6 *The hedge-based valuation $\rho^{HB}[S]$ for the claim S given by (5.16) can be expressed as follows*

$$\rho^{HB}[S] = \mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \beta\sqrt{\mathbb{E}\left[\left(S^h\right)^2\right]}\sqrt{\mathbb{E}\left[\frac{Var[g_1 | \mathcal{F}^{f,s}]}{n^a}\right] + Var[\mathbb{E}[g_1 | \boldsymbol{Z}]]}. \quad (5.19)$$

The 2-step valuation $\rho^{2-step}[S]$ for the claim S given by (5.16) can be expressed as follows

$$\rho^{2-step}[S] = \mathbb{E}_{\mathbb{Q}}\left[H_S^h\right] + \beta\mathbb{E}_{\mathbb{Q}}\left[S^h\right]\sqrt{\mathbb{E}\left[\frac{Var[g_1 | \mathcal{F}^{f,s}]}{n^a}\right] + Var[\mathbb{E}[g_1 | \boldsymbol{Z}]]}. \quad (5.20)$$

The hedge-based valuation and the 2-step valuation are combining an actuarial valuation with a financial valuation, but do not distinguish between non-traded systematic and unsystematic risks. Similar to the 3-step valuations (5.17) and (5.18), the hedge-based valuation (5.19) and the 2-step valuation (5.20) start with the price of the hedgeable claim. The hedge-based and 2-step valuation use the actuarial valuation for diversifiable and systematic risks. The term $\mathbb{E}\left[\frac{\text{Var}\left[g_1 | \mathcal{F}^{f,s}\right]}{n^a}\right]$ corresponds with the unsystematic risk, whereas $\text{Var}\left[\mathbb{E}\left[g_1 | \boldsymbol{Z}\right]\right]$ captures the systematic deviations. The difference between the 2-step and the hedge-based valuation lies in the valuation of the financial risk present in the non-hedgeable part of the claim. The hedge-based valuation considers the financial risk in the non-hedgeable part as a systematic risk and therefore uses the actuarial valuation. The 2-step valuation, on the other hand, uses a risk neutral valuation for the financial risk of the non-hedgeable part. Note that we have the flexibility in the 3-step valuation to either use risk neutral or real world valuation for the financial risk in the non-hedgeable part of the claim, by tuning the systematic valuation in an appropriate way.

Assume the actuarial risks $\boldsymbol{X}$ are also independent of the systematic risks. In this situation, the claim S given by (5.16) is a product claim which only depends on the financial risks through the claim S^h and the independent risks $\boldsymbol{X}$ through the actuarial payoff functions g_i. Since $\mathbb{E}\left[g_1 | \boldsymbol{Z}\right] - \mathbb{E}[g_1] = 0$, We can determine the 3-step and the additive valuation for this particular situation:

$$\begin{aligned}
\rho^{3-step}[S] &= \mathbb{E}_{\mathbb{Q}}\left[S^h\right]\mathbb{E}\left[g_1\right] + \rho^s\left[S^h\right]\beta\sqrt{\frac{\text{Var}\left[g_1\right]}{n^a}}. \\
\rho^{+}[S] &= \mathbb{E}_{\mathbb{Q}}\left[S^h\right]\mathbb{E}\left[g_1\right] + \sqrt{\mathbb{E}\left[\left(S^h\right)^2\right]}\beta\sqrt{\frac{\text{Var}\left[g_1\right]}{n^a}}.
\end{aligned}$$

We find that in case there are no-systematic risks, the additive 3-step valuation and the hedge-based valuation coincide. If the systematic valuation ρ^s is such that it is consistent with the risk neutral measure, we find that the 3-step valuation coincides with the 2-step valuation.

6 Conclusion

In this paper we introduced a new class of market-consistent valuations, which we called the 3-step hedge-based valuations, for the valuation of hybrid claims that depend on financial, actuarial and systematic risks. This new valuation principle uses traded assets to construct a hedging portfolio for the hybrid liability. The value of this portfolio can be determined using the observable market prices. Equivalently, we express this price as a risk neutral expectation. Unlike financial risks, actuarial and systematic risks are not traded and therefore the hedging portfolio will not perfectly replicate the hybrid claim. We then employ a 2-step valuation for the residual, unhedgeable, part of the claim. The 3-step hedge-based valuation is a market-consistent valuation and can therefore be used to determine regulatory capital for complex insurance liabilities. Moreover, our valuation takes into account that systematic risks are fundamentally different from actuarial risks and therefore require a different valuation principle.

The hedge-based valuation introduced in Dhaene et al. (2017) and the 2-step valuation introduced in Pelsser and Stadje (2014) use market prices to value the hedgeable part of a claim whereas a model valuation is used to model the residual part of the claim. We show that a 3-step hedge-based valuation can be defined by specifying the model valuation in such a way that it distinguishes between the systematic part and the actuarial part of the non-hedgeable part of the claim. Therefore, the 3-step hedge-based valuations compose a subclass of the hedge-based valuations and the 2-step valuations.

The idea of a fair valuation was introduced in Dhaene et al. (2017). A fair valuation finds a balance between pricing through hedging and pricing through modelling. Hedgeable claims should be priced using the replicating portfolio. Since we can observe the prices of the traded assets, pricing through replication is model free. Market-consistency of the valuation ensures that while valuating a hybrid claim, the hedgeable part is always consistent with the observable prices. However, the incompleteness of the market requires a model to assess the risks of the unhedgeable part of the claim.The underlying model should be such that orthogonal systematic claims are priced using the systematic valuation and orthogonal actuarial claims are priced using the actuarial valuation. We show that the 3-step hedge-based valuation is a fair valuation. Moreover, under a linearity condition for the systematic valuation, we show that the class of fair valuations coincides with the class of 3-step hedge-based valuations.

We follow Dhaene et al. (2017) and Deelstra et al. (2020) to decompose a hybrid claim in a hedgeable, a systematic and an actuarial part. We then define an additive 3-step valuation by applying an appropriate valuation to each of these parts. We show that this valuation is similar to the valuation defined in Deelstra et al. (2020). Moreover, we show that the additive 3-step valuation is a 3-step hedge-based valuation. In order to understand the difference between the 3-step hedge-based valuation, the additive 3-step and other existing market-consistent valuations, we consider a product claim consisting of a hedgeable part and an actuarial part. The actuarial risks are assumed to be conditionally independent. We derive closed-form expressions for the different valuations. The main advantage of the 3-step valuation is that it allows for dependencies between the systematic and unsystematic parts of the residual claim. Moreover, the systematic risk can be valuated using a different valuation than the actuarial valuation.

In Section 4.3 and 5.2 we considered two simple examples to illustrate the 3-step hedge-based valuation and the additive 3-step hedge-based valuation. For mathematical convenience and to be able to focus on the intuition of the newly-proposed valuations, we imposed several simplifying assumptions. For example, in Section 4.3, we assumed stock markets and inflation to be independent. Moreover, all risks are assumed to be 2-point distributions. In a future research topic, we will further investigate this type of product claims, but relax the independence assumption, consider more realistic distributions and use available data to calibrate the distributions. The example considered in Section 5.2 derives the closed-form solutions for the different valuations discussed in this paper. In a future research topic, we will model and calibrate the financial, actuarial and systematic risk factors using available data; see e.g. Deelstra et al. (2020). This will enable us to numerically investigate the differences between the 3-step hedge-based and the additive 3-step valuation.

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A Appendix : proofs

A.1 Proof of Expression

We start from Expression (4.17). We can then write

$$\begin{aligned}\rho^{3-\text{step}}[S] &= \mathbb{E}_{\mathbb{Q}}[Y_1]\,\mathbb{E}_{\mathbb{P}}[XZ] + \rho^s\left[Y_1\mathbb{E}_{\mathbb{P}}[X]\,(Z-\mathbb{E}_{\mathbb{P}}[Z]) + Y_1 Z\beta\sqrt{\text{Var}_{\mathbb{P}}[X]}\right]\\ &= \mathbb{E}_{\mathbb{Q}}[Y_1]\,\mathbb{E}_{\mathbb{P}}[XZ] + \mathbb{E}\left[\varphi^f\varphi^s Y_1\mathbb{E}_{\mathbb{P}}[X]\,(Z-\mathbb{E}_{\mathbb{P}}[Z])\right] + \mathbb{E}\left[\varphi^f\varphi^s Y_1 Z\beta\sqrt{\text{Var}_{\mathbb{P}}[X]}\right]\\ &= \mathbb{E}_{\mathbb{Q}}[Y_1]\,\mathbb{E}_{\mathbb{P}}[XZ] + \mathbb{E}_{\mathbb{P}}[X]\,\mathbb{E}\left[\varphi^f\varphi^s Y_1 Z\right] - \mathbb{E}_{\mathbb{P}}[X]\,\mathbb{E}_{\mathbb{P}}[Z]\,\mathbb{E}\left[\varphi^f\varphi^s Y_1\right]\\ &\quad + \beta\sqrt{\text{Var}_{\mathbb{P}}[X]}\,\mathbb{E}\left[\varphi^f\varphi^s Y_1 Z\right].\end{aligned}$$

Note that φ^s is $\mathcal{F}^s$-measurable, φ^f is $\mathcal{F}^f$-measurable and $\mathbb{E}\left[\varphi^f Y_i\right] = \mathbb{E}_{\mathbb{Q}}[Y_1]$. If we use the independence between $\boldsymbol{Y}$ and $\boldsymbol{Z}$, we find

$$\rho^{3-\text{step}}[S] = \mathbb{E}_{\mathbb{P}}[X]\,\mathbb{E}_{\mathbb{Q}}[Y_1]\,\mathbb{E}[\varphi^s Z] + \beta\sqrt{\text{Var}_{\mathbb{P}}[X]}\,\mathbb{E}\left[\varphi^f\varphi^s Y_1 Z\right],$$

which proves the result.

A.2 Proof of Expressions (4.13) and (4.14)

We have that the conditional probability $\mathbb{P}[X=1|\,Z=z_1]$ is equal to α; see (4.12). W can write

$$\mathbb{P}[X=1] = \mathbb{P}[X=1|\,Z=z_1]\,\frac{1}{2} + \mathbb{P}[X=1|\,Z=z_2]\,\frac{1}{2}.$$

From (4.8), we then find

$$\mathbb{P}\left[X=1 \mid Z=z_2\right]=1-\alpha.$$

Using the conditional probabilities, we can determine the conditional expectation $\mathbb{E}_{\mathbb{P}}\left[X \mid Z=z_1\right]$ as follows

$$\mathbb{E}_{\mathbb{P}}\left[X \mid Z=z_1\right]=1 \times \mathbb{P}\left[X=1 \mid Z=z_1\right]=\alpha.$$

Similarly, we have that

$$\mathbb{E}_{\mathbb{P}}\left[X \mid Z=z_2\right] = 1 \times \mathbb{P}\left[X=1 \mid Z=z_z\right]=1-\alpha$$

We can then write the following

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[X \mid Z] &= I \alpha+(1-I)(1-\alpha) \\ &= I(2 \alpha-1)+(1-\alpha). \end{aligned}$$

One can verify that the conditional variance $\operatorname{Var}_{\mathbb{P}}[X \mid Z]$ is deterministic. Indeed, we have that

$$\operatorname{Var}_{\mathbb{P}}\left[X \mid Z=z_1\right]=\operatorname{Var}_{\mathbb{P}}\left[X \mid Z=z_2\right]=\alpha(1-\alpha),$$

from which we find:

$$\operatorname{Var}_{\mathbb{P}}[X \mid Z]=\alpha(1-\alpha).$$

A.3 Proof of Expression (4.15)

The 3-step valuation is given as follows

$$\rho^{3-\text{step}}[S]=\theta_S^{(1)} y_1+\rho^s\left[\rho^a\left[S-\theta_S^{(1)} Y_1 \mid \mathcal{F}^{f, s}\right]\right].$$

Since $S=X Y_1 Z$ and using the conditionally homogeneity and translation invariance of the conditional valuation of ρ^a, we can write the following

$$\begin{aligned} \rho^{3-\text{step}}[S] &= \theta_S^{(1)} y_1+\rho^s\left[\rho^a\left[Y_1\left(X Z-\theta_S^{(1)}\right) \mid \mathcal{F}^{f, s}\right]\right] \\ &= \theta_S^{(1)} y_1+\rho^s\left[Y_1\left(Z \rho^a\left[X \mid \mathcal{F}^{f, s}\right]-\theta_S^{(1)}\right)\right] \end{aligned}$$

The conditional valuation ρ^a is the conditional standard deviation principle with risk loading β. Using (4.13) and (4.14) then proves Expression (4.15).

A.4 Proof of Proposition 4

Note that $S-H_S^h=H_S^a+H_S^s$. We can then rewrite $\rho^s\left[\rho^a\left[H_S^a\right] \mid \mathcal{F}^{f, s}\right]$ as follows

$$\begin{aligned} \rho^a\left[H_S^a \mid \mathcal{F}^{f, s}\right] &= \rho^a\left[S^h\left(\frac{\sum_{i=1}^{n^{(a)}} g_i}{n^{(a)}}-\mathbb{E}\left[g_1 \mid \boldsymbol{Z}\right]\right) \mid \mathcal{F}^{f, s}\right] \\ &= S^h\left(\rho^a\left[\frac{\sum_{i=1}^{n^{(a)}} g_i}{n^{(a)}} \mid \mathcal{F}^{f, s}\right]-\mathbb{E}\left[g_1 \mid \boldsymbol{Z}\right]\right). \end{aligned}$$

If we take the conditional standard deviation principle, we find:

$$\begin{aligned}
\rho^a\left[S - H_S^h \middle| \mathcal{F}^{f,s}\right] &= \rho^a\left[H_S^a + H_S^s \middle| \mathcal{F}^{f,s}\right] \\
&= S^h\left(\mathbb{E}\left[\frac{\sum_{i=1}^{n^{(a)}} g_i}{n^{(a)}}\middle| \mathcal{F}^{f,s}\right] + \beta\sqrt{\mathrm{Var}\left[\frac{\sum_{i=1}^{n^{(a)}} g_i}{n^{(a)}}\middle| \mathcal{F}^{f,s}\right]} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right) + H_S^s \\
&= S^h\beta\sqrt{\frac{\mathrm{Var}\left[g_1 \middle| \boldsymbol{Z}\right]}{n^a}} + S^h\left(\mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right] - \mathbb{E}\left[g_1\right]\right) \\
&= S^h\left(\mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right] - \mathbb{E}\left[g_1\right] + \beta\sqrt{\frac{\mathrm{Var}\left[g_1 \middle| \boldsymbol{Z}\right]}{n^a}}\right).
\end{aligned}$$

Using Expression (4.4) of the 3-step hedge-based valuation proves the result.

A.5 Proof of Proposition 5

Proof. We have that

$$\rho^s\left[H_S^s\right] = \rho^s\left[S^h\left(\mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right] - \mathbb{E}\left[g_1\right]\right)\right]. \tag{A.1}$$

For the valuation of the actuarial part, we find

$$\rho^s\left[H_S^a\right] = \mathbb{E}\left[S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right)\right] + \beta\sqrt{\mathrm{Var}\left[S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right)\right]}, \tag{A.2}$$

where we assumed that the actuarial valuation is the standard deviation principle. Using the independence between the financial risks and the systematic and actuarial risks, we find that the first term in (A.2) is zero. The variance in (A.2) can be expressed as follows

$$\begin{aligned}
\mathrm{Var}\left[S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right)\right] &= \mathbb{E}\left[\mathrm{Var}\left[S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right) \middle| \mathcal{F}^{f,s}\right]\right] \\
&\quad + \mathrm{Var}\left[\mathbb{E}\left[S^h\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a} - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right) \middle| \mathcal{F}^{f,s}\right]\right] \\
&= \mathbb{E}\left[\left(S^h\right)^2 \mathrm{Var}\left[\left(\frac{\sum_{i=1}^{n^a} g_i}{n^a}\right) \middle| \mathcal{F}^{f,s}\right]\right] \\
&\quad + \mathrm{Var}\left[S^h\left(\mathbb{E}\left[g_1 \middle| \mathcal{F}^{f,s}\right] - \mathbb{E}\left[g_1 \middle| \boldsymbol{Z}\right]\right)\right] \\
&= \mathbb{E}\left[\left(S^h\right)^2\right]\mathbb{E}\left[\frac{\mathrm{Var}\left[g_1 \middle| \mathcal{F}^{f,s}\right]}{n^a}\right].
\end{aligned} \tag{A.3}$$

Combining (A.3) and (A.1) gives the desired result. ∎

A.6 Proof of Proposition 6

For a proof of Expression (5.19), we refer to Exercise 12 in Dhaene (2020). In order to prove Expression (5.20), we write:

$$
\begin{aligned}
\rho^{2-\text{step}}[S] &= \mathbb{E}_{\mathbb{Q}}\left[\rho^a\left[S|\,\mathcal{F}^f\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[S^h\rho^a\left[\frac{1}{n^a}\sum_{i=1}^{n^a} g_i|\,\mathcal{F}^f\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[S^h\right]\rho^a\left[\frac{1}{n^a}\sum_{i=1}^{n^a} g_i\right], \qquad \text{(A.4)}
\end{aligned}
$$

where we used $S^h > 0$ in the first step and the independence between the financial risks and the actuarial risks. If ρ^a is the standard deviation principle, we find:

$$
\rho^a\left[\frac{1}{n^a}\sum_{i=1}^{n^a} g_i\right] = \mathbb{E}\left[g_1\right] + \frac{\mathbb{E}\left[\text{Var}\left[g_1|\,\boldsymbol{Z}\right]\right]}{n^a} + \text{Var}\left[\mathbb{E}\left[g_1|\,\boldsymbol{Z}\right]\right].
$$

Plugging this expression in (A.4) proves the result.