1 Exponential utility and insurance premium

An insurer having initial wealth $W$ and utility function $u$ is said to have exponential utility if the function $u$ can be expressed as follows:

$$u(x) = -\alpha e^{-\alpha x}.$$ 

Consider a non-negative risk $X$. We first determine the minimal premium $P$ for which the insurer is willing to insure the risk $X$. This premium will be a function of the parameter $\alpha$ and we denote it as $P_X(\alpha)$. It is the solution of the following equation:

$$u(W) = \mathbb{E}[u(W + P - X)].$$

This equation can be solved for $P$ as follows:

$$u(W) = \mathbb{E}[u(W + P - X)]$$

$$\leftrightarrow -\alpha e^{-\alpha W} = \mathbb{E}[-\alpha e^{-\alpha(W+P-X)}]$$

$$\leftrightarrow e^{-\alpha W} = e^{-\alpha W} e^{-\alpha P} \mathbb{E}[e^{\alpha X}]$$

$$\leftrightarrow e^{\alpha P} = \mathbb{E}[e^{\alpha X}].$$

We find the following premium $P_X(\alpha)$ for the risk $X$:

$$P_X(\alpha) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X}].$$

This premium is independent of the initial wealth $W$.

The risk-aversion for an insurer having exponential utility with parameter $\alpha$ is constant and given by:

$$r(x) = \frac{-u''(x)}{u'(x)} = \frac{\alpha^3 e^{-\alpha x}}{\alpha^2 e^{-\alpha x}} = \alpha.$$
2 Risk aversion and the insurance premium

The insurance premium $P_X(\alpha)$ is increasing in $\alpha$. This can be proven as follows. Consider $0 < \alpha < \gamma$. Define the function $v$ as $v(x) = x^{\alpha/\gamma}$. One can verify in a straightforward way that $v'(x) = \frac{\alpha}{\gamma} x^{\alpha/\gamma - 1}$ and $v''(x) = \frac{\alpha}{\gamma^2} \left( \frac{\alpha}{\gamma} - 1 \right) x^{\alpha/\gamma - 2}$. Then we find that $v''(x) < 0$ and thus $v$ is strictly concave. From Jensen’s inequality, we find:

$$\mathbb{E}[v(Y)] < v(\mathbb{E}[Y]).$$

Define $Y = e^{\gamma X}$. Then $v(Y) = (e^{\gamma X})^{\alpha/\gamma} = e^{\alpha X}$. We can then prove the following inequality:

$$(\mathbb{E}[e^{\gamma X}])^\alpha = \left( \mathbb{E} \left[ \frac{e^{\gamma X}}{\mathbb{E}[Y]} \right]^{\alpha/\gamma} \right)^\gamma = v(\mathbb{E}[Y])^\gamma > \mathbb{E}[v(Y)]^\gamma = (\mathbb{E}[e^{\alpha X}])^\gamma.$$

Taking the logarithms on both sides results in the following inequality:

$$P_X(\alpha) < P_X(\gamma).$$

We conclude that a higher risk aversion parameter results in a higher premium, i.e. the insurer requires a higher premium in order to take over the risk $X$ from the insured.

Take $\alpha$ very small and remember the following Taylor expansions:

$$e^{\alpha x} = 1 + \alpha x + \ldots$$
$$\log(1 + x) = x + \ldots.$$

The premium $P_X(\alpha)$ can be approximated as follows:

$$P_X(\alpha) \approx \frac{1}{\alpha} \log (1 + \alpha \mathbb{E}[X]) \approx \frac{1}{\alpha} \alpha \mathbb{E}[X] = \mathbb{E}[X].$$

We conclude:

$$\lim_{\alpha \to 0} P_X(\alpha) = \mathbb{E}[X].$$

The insurance premium is increasing in the risk aversion. Hence, the expectation $\mathbb{E}[X]$ is always a lower bound for the insurance premium. The insurer is willing to take over the risk $X$ for a premium equal to the expectation if he has no risk aversion, i.e. the insurer is risk neutral.
3 Aggregating exponential premiums

Consider an insurer with exponential utility function \( u(x) = -\alpha e^{-\alpha x} \). The risks \( X_1, X_2, \ldots, X_n \) are independent and they all have the same distribution as the r.v. \( X \). So \( X_i \overset{d}{=} X \), for each \( i = 1, 2, \ldots, n \). The aggregated loss \( S \) is equal to

\[
S = X_1 + X_2 + \ldots + X_n.
\]

The minimal premium the insurer wants to receive for insuring \( S \) is denoted by \( P_S(\alpha) \). This premium \( P_S(\alpha) \) satisfies the equation

\[
\mathbb{E}[u(W + P_S(\alpha) - S)] = u(W).
\]

We find that

\[
P_S(\alpha) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha S}].
\]

If we use that the \( X_i \) are independent, we can write

\[
P_S(\alpha) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha(X_1 + X_2 + \ldots + X_n)}] = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X_1}] \mathbb{E}[e^{\alpha X_2}] \ldots \mathbb{E}[e^{\alpha X_n}] = \frac{1}{\alpha} \log \left( \mathbb{E}[e^{\alpha X_1}] + \log \mathbb{E}[e^{\alpha X_2}] + \ldots + \log \mathbb{E}[e^{\alpha X_n}] \right) = \frac{1}{\alpha} \left( \log \mathbb{E}[e^{\alpha X_1}] + \log \mathbb{E}[e^{\alpha X_2}] + \ldots + \log \mathbb{E}[e^{\alpha X_n}] \right) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X}] = n \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha X}] = n P_X(\alpha)
\]

If each insured pays his own premium \( P_X(\alpha) \) for insuring his risk \( X_i \), the insurer collects enough money to cover the insurance premium for \( S \).

Define the r.v. \( Y \) as

\[
Y = \frac{\sum_{i=1}^{n} X_i}{n}.
\]

The premium \( P_Y(\alpha) \) is given by

\[
P_Y(\alpha) = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha Y}] = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha \frac{\sum_{i=1}^{n} X_i}{n}}] = \frac{1}{\alpha} \log \mathbb{E}[e^{\alpha \frac{X_1}{n} + \alpha \frac{X_2}{n} + \ldots + \alpha \frac{X_n}{n}}] = \frac{1}{\alpha} \log \left( \mathbb{E}[e^{\alpha X_1}] \mathbb{E}[e^{\alpha X_2}] \ldots \mathbb{E}[e^{\alpha X_n}] \right) = \frac{1}{\alpha} \log \mathbb{E}[e^{\frac{\alpha}{n} X}] = P_X\left(\frac{\alpha}{n}\right).
\]
Note that the premium $P_X(\alpha)$ is increasing in $\alpha$, which means that
\[ \mathbb{E}[X] \leq P_Y(\alpha) \leq P_X(\alpha). \]

The insurer can aggregate all the losses and divide it equally over its policy holders. Then each policy holder pays the loss $Y$. Therefore, it is sufficient to ask the premium $P_Y(\alpha)$. If $n$ is large, $\alpha/n$ tends to zero. Taking into account $\mathbb{E}[Y] = \mathbb{E}[X]$, we find that $P_Y(\alpha) \approx \mathbb{E}[X]$, if $n$ is large.

The random vector $(X_1, X_2, \ldots, X_n)$ contains independent random variables. Knowledge about the realization of the first risk $X_1$, does not give any information about the realization for $X_2$. The random vector $(Y_1, Y_2, \ldots, Y_n)$, where $Y_i = Y$, contains random variables which are dependent. Moreover, they are extremely positive dependent in that knowing $Y_1$ means the realization of all other random variables is known. Because the random losses $Y_i$ are not independent anymore, one cannot add them to determine the aggregate premium.