Exercises: Foundations of Quantitative Risk Measurement
Chapter 1: Expected Utility Theory

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1. Consider an insured who is facing a risk X with mean μ and variance σ^2. The insured is a utility maximizer and has a concave utility function u. Show that the maximal premium P^M the insured is willing to pay to insure the risk X, can be approximated as follows:

$$P^M \approx \mu + \frac{1}{2} r\left(w-\mu\right)\sigma^2, \tag{0.1}$$

   where the function r is the absolute risk aversion.

   (*Hint: use a Taylor expansion for the expected utility.*)

2. Consider an insurer with a total wealth $W = 100000$ is holding future aggregate liabilities denoted by X. The probability function of X is given by:

$$\mathbb{P}\left[X = k\right] = \frac{1}{3}, \text{ for } k \in \{0, 1000, 100000\}.$$

   The value of these future liabilities is equal to the maximal premium the insurer is willing to pay to reinsurance the total risk X, assuming the insurer acts as a utility maximizer.

   (a) Assume the utility function of the insurer is given by $U(x) = \sqrt{x}$. Determine the value of the future liabilities, denoted by V.

   (b) Assume the insurer can, by implementing an efficient risk management program, reduce the extreme risks. Instead of having a maximal loss of 100 000, the maximal loss is now 80 000. The new liability is denoted by Y and given by

$$\mathbb{P}\left[X = k\right] = \frac{1}{3}, \text{ for } k \in \{0, 1000, 80000\}.$$

   Determine the value V_Y of this new liability.

   (c) An optimalization of the risk management program further reduces the extreme risks. The new liability Z of the insurance company has now a maximal loss of 60 000. Determine the value V_Z of this new liability.

   (d) Compare the ratios $\frac{V_Y}{V}$ and $\frac{V_Z}{V}$. What can you conclude?

3. Consider an insurance contract which pays 100000 euro's within 20 years if the insured (x) is still alive at this moment. In the other situation, there is no payment. The probability that (x) survives for 20 years is $p = \frac{3}{4}$.
The insurer has the possibility to invest the earned premiums in a bank account which pays a compound interest of 3%.
   (a) Assume the insurer is a utility maximizer with utility function given by $U_1(x) = x$ and zero initial wealth. Calculate the minimal premium P_1 the insurer wants to receive when selling this contract. This premium is paid at the beginning of the contract, whereas the benefit is paid after 20 years.
   (Remark: The payment of the premium and the payment of the loss happen in different periods in time. It is only meaningful to compare utilities at the same period in time.)
   (b) Assume the utility function of the insurer is given by $U_2(x) = -\alpha \mathrm{e}^{-\alpha x}$, where $\alpha = 0.001$. Determine the minimal premium P_2 the insurer wants to receive in this case.
   (c) Determine the Arrow-Pratt measure of absolute risk-aversion for the utility function U_2.
   (d) If α is not specified, P_2 depends on this choice of α. Therefore, we denote this premium by $P_2(\alpha)$. Assume that α is tending towards zero. Use Excel (or MatLab, R, ...) to determine the premium in this situation. Give an interpretation.
   (e) (Extra) Prove that $P_2(\alpha) \approx P_1$ if $\alpha \approx 0$.
   (*Hint: Use a first order approximation*)

4. Consider an investor, with initial wealth w and utility function $u(x) = \log x$. The investor has the possibility to invest his money in a risky asset for a duration of one year. The return [1] after one year is random and denoted by X. This yearly return will be e^{σ} or $\mathrm{e}^{-\sigma}$. Both outcomes happen with probability $\frac{1}{2}$. Assume that $w = 100$ and $\sigma = 0.2$.
   (a) Assume the investor can choose between keeping the money in his pocket or investing this amount in X. Which choice will be more preferable to the decision maker if he is a utility maximizer?
   (b) Suppose he can invest the proportion α of his wealth in the risky asset. This value α can be larger than 1, meaning that the investor takes a short position, which we assume he can do for free. If the expected utility after one year of this investment is denoted by $f(\alpha)$, prove that
$$f(\alpha) = \log(w) + \frac{1}{2}\log(\alpha(\mathrm{e}^{\sigma} - 1) + 1) + \frac{1}{2}\log\left(\alpha\left(\mathrm{e}^{-\sigma} - 1\right) + 1\right).$$
   Use Excel (or MatLab, R,...) to plot the function f. Give an interpretation.
   (c) The amount $c(\alpha)$ is defined as the amount such that the investor is indifferent between receiving the fixed amount $c(\alpha)$ and investing a proportion α of his wealth in the asset X. Determine $c(\alpha)$. For which α is $c(\alpha)$ maximal? Give an interpretation of this result.

5. Consider an insurer with exponential utility function $U(x) = -\alpha \mathrm{e}^{-\alpha x}$. The risks $X_1, X_2, \ldots, X_n$ are independent and they all have the same distribution as the r.v. X. So $X_i \stackrel{\mathrm{d}}{=} X$, for each $i = 1, 2, \ldots, n$. The aggregated loss S is equal to
$$S = X_1 + X_2 + \ldots + X_n.$$

[1] if an asset has a return equal to X, an investment w will be worth wX

(a) Prove that the minimal premium the insurer asks for insuring the risk X is given by

$$P_X(\alpha) = \frac{1}{\alpha} \log \mathbb{E}\left[\mathrm{e}^{\alpha X}\right].$$

Is this premium increasing or decreasing in function of α?

(b) The minimal premium the insurer wants to receive for insuring S is denoted by $P_S(\alpha)$. Prove that:

$$P_S(\alpha) = n P_X(\alpha).$$

Give an interpretation.

(c) Define the r.v. Y as

$$Y = \frac{\sum_{i=1}^{n} X_i}{n}.$$

Prove that

$$P_Y(\alpha) = P_X\left(\frac{\alpha}{n}\right).$$

1 Previous Exam Questions

1. Assume that the stop-loss transform of a loss X is equal to

$$\pi_X(x) = 50e^{-0.02x}, \qquad \text{for } x > 0.$$

(a) Calculate the distribution function F_X and the density function f_X for this loss X.

(b) An individual wants to insure himself against this loss X. Calculate the maximal premium that he is willing to pay to the insurance company, if his utility function is equal to

$$u(x) = -0.01e^{-0.01x}.$$

(c) Now assume that the insurer has a utility function given by

$$U(x) = -(2W - x)^2,$$

where W stands for the capital of the insurer. Assume further that the insurer wants a minimal premium of 60 to be paid by the individual. Calculate the capital W of the insurer.