Foundations of Quantitative Risk Measurement

Chapter 1: Expected Utility Theory[1]

[1]Chapter 1 from 'Foundations of Quantitative Risk Measurement', Dhaene, J., Denuit, M., Linders, D. and Verdonck, T. To be published.

0 – Outline

- Examples of decision making problems:
  - *Individual*: bear a risk or insure it (partially)?
  - *Insurer*: accept a risk or not?
  - *Insurer*: reinsure (part of) the accepted risks?
- Optimal choice of the decision maker depends on:
  - his initial wealth,
  - his risk appetite.

- Theories of choice under risk:
  - Expected utility theory: Cramer (1728), Bernouilli 1738), Von Neumann & Morgenstern (1947).
  - Dual theory of choice under risk: Yaari (1987), Roëll (1987), Schmeidler (1989).
- Common properties of these theories:
  - Preference relations of a decision maker are qualitative in nature,
  - but follow from comparing numerical quantities.

Conventions and definitions

- f is a real-valued function.

$$f : I \to \mathbb{R},$$

  - I is an interval in $\mathbb{R}$.
    - $I = [a,b], [a,+\infty), (-\infty,b]$ or $(-\infty,+\infty)$.
  - f is a real-valued function.
- Non-decreasing functions:
  - The function f is non-decreasing if the following holds

$$x \leq y \to f(x) \leq f(y), \quad \text{for } x,y \in \mathbb{R}.$$

  - f is non-increasing if $-f$ is non-decreasing.

Conventions and definitions

- Continuous functions:
  - The function f is continuous on $[a,b]$ if it is
    - continuous on (a,b);
    - right continuous at a;
    - left continuous at b.
- Differentiable functions:
  - The function f is differentiable on $[a,b]$ if it is
    - differentiable on (a,b);
    - differentiable from the right continuous at a;
    - differentiable from the left at b.

Random variables and cumulative distribution functions

- Probability space: $(\Omega, \mathcal{F}, \mathbb{P})$
  - Ω : all future states of nature.
    - $\omega \in \Omega$: a possible realization of a random experiment.
  - $\mathcal{F}$: set of events.
    - Collection of subsets of Ω.
  - $\mathbb{P}$: probability measure assigning the likelihood to each event.
    - Attach a probability to each event.
    - $\mathbb{P} : \mathcal{F} \to [0,1]$.
    - $\mathbb{P}[\Omega] = 1$.

Random variables and cumulative distribution functions

- A random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *function* which attaches a *real number* to each possible outcome:

$$X : \Omega \longrightarrow \mathbb{R}.$$

  - ω describes the state of a random phenomenon.
  - $X[\omega]$ is a single aspect of the state ω.

- Define the set $X^{-1}(I)$ as:

$$X^{-1}(I) = \{\omega \in \Omega |\ X[\omega] \in I\}, \text{ where } I \subseteq \mathbb{R}.$$

  - Assumption: The random variable X is a measurable function on the measurable space $(\Omega, \mathcal{F})$. Hence: $X^{-1}(I) \in \mathcal{F}$.

Random variables and cumulative distribution functions

- What is the probability that X lies in the interval B?
  - Probability function $\mathbb{P}$ assigns probabilities to subsets of Ω.
  - $\mathbb{P}\left[X^{-1}(B)\right]$ = probability that X takes a value in B.
  - Notation:

$$\mathbb{P}\left[X \in B\right] = \mathbb{P}\left[X^{-1}(B)\right].$$

- We assume that the probability $\mathbb{P}[X \in B]$ is known.
- *The only uncertainty when considering a future random loss is the uncertainty about its particular future outcome, not the uncertainty about its 'distribution'.*

Random variables and cumulative distribution functions

- Cumulative distribution function (cdf) F_X of the r.v. X :

$$F_X(x) = \mathbb{P}[X \leq x], \text{ for } x \in \mathbb{R}.$$

  - F_X is non-decreasing and right continuous.

- Assume F_X has a jump of size $\Delta(x)$ in x :
  - $\Delta(x) = F_X(x) - F_X(x-)$.
  - $\Delta(x)$ is zero if F_X is continuous in x.
  - For all $x \in \mathbb{R}$:

$$\mathbb{P}[X = x] = \Delta(x).$$

Random variables and cumulative distribution functions

- The expectation:

$$\mathbb{E}\left[g(X)\right] = \int_I g(x) \mathrm{d}F_X\left(x\right).$$

  - $X : \Omega \to I$ and $g : I \to \mathbb{R}$.

- Assumption:
  - The expectation of X is finite:

$$\mathbb{E}[X] < \infty.$$

  - Expressions involving $\mathbb{E}\left[g(X)\right]$ are provided this quantity is well-defined and finite.

Random variables and cumulative distribution functions

- <u>If F_X has only a discrete part:</u>

$$\mathbb{E}\left[g(X)\right] = \sum_y g(y)\Delta\left(y\right) = \sum_y g(y)\mathbb{P}\left[X = y\right].$$

- <u>If F_X has a discrete and continuous part:</u>

$$\mathbb{E}\left[g(X)\right] = \int_I g(x) f_X\left(x\right) dx + \sum_y g(y)\Delta\left(y\right).$$

  - $f_X\left(x\right)\mathrm{d}x =$ probability that X takes a value in the $\left[x, x + \mathrm{d}x\right]$.

The St. Petersburg Paradox

- Problem:
  - *A fair coin is tossed repeatedly until it lands head up. The income you receive is equal to 2^n if the first head appears on the n-th toss. How much are you willing to pay for this game?*
- Expected gain:
  - Assume that the coin is fair.
  - Probability to win the amount 2^n is $\frac{1}{2^n}$.
  - The expected gain:

$$\sum_{n=1}^{+\infty}(2^n) \times \frac{1}{2^n} = \sum_{n=1}^{+\infty} 1 = +\infty.$$

The St. Petersburg Paradox

- The expected income from this game is $+\infty$, however ...
  - A decision maker will not pay $+\infty$.
  - The price to play this game will be modest.
- Conclusion:
  - The expectation is not (always) a good method to rank uncertain outcomes.

The St. Petersburg Paradox

- Classical expected utility theory:
  - Each decision maker assigns a utility $u(x)$ to any fortune of amount x.
  - Utility functions are of a subjective nature.
  - 'Reasonable' utility functions share common properties:
    - non-decreasingness,
    - decreasing marginal utility.
- Expected utility and insurance:
  - Why is an individual willing to pay a premium larger than the average expected loss?
  - Why are certain insurance covers to be preferred over others?
  - Behavior of insureds:
    - moral hazard,
    - anti-selection.

The St. Petersburg Paradox

- Consider a decision maker with initial fortune w.
- He attaches a utility $u(x)$ to a fortune x.
- The price to play the game is P.
- Assume our agent wins after n throws.
  - His utility if he wins after n throws: $u\left(w-P+2^{n}\right)$.
  - Probability to win after n throws: $\frac{1}{2^{n}}$.
- Expected utility:
  - At initiation, the utility he will reach if he plays the game is **unknown**.
  - Expected utility:

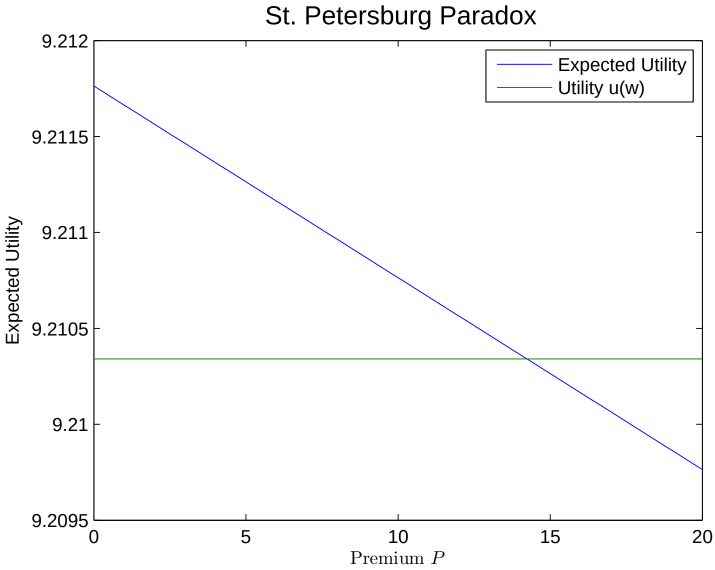
$$\sum_{n=1}^{+\infty} u\left(w-P+2^{n}\right)\frac{1}{2^{n}}.$$

The St. Petersburg Paradox

- The decision maker is an *expected utility maximizer*.
- If he doesn't play the game, his utility is $u(w)$.
- He is willing to play the coin tossing game for a price P if

$$u(w) \leq \sum_{n=1}^{+\infty} u\left(w - P + 2^n\right) \frac{1}{2^n}$$

  - G. Cramer: $u(x) = \sqrt{x}$.
  - D. Bernoulli: $u(x) = \ln x$.
- Example:
  - Take $w = 10000$ and $u(x) = \ln x$.
  - Then $P = 14.2385$. (Check this using MatLab or Excel!)


St. Petersburg Paradox
Expected Utility
Utility u(w)
9.212
9.2115
9.211
9.2105
9.21
9.2095
0
5
10
15
20
Expected Utility
Premium P

Transformed wealth levels

Definition (Utility function)

A **utility function** u is a real-valued *non-decreasing* function asserting a decision maker's utility-of-wealth $u(x)$ to each possible level of wealth x.

- Decision makers have *non-negative marginal utility*: more wealth is always preferred over less wealth.
- In general, different decision makers will have different utility functions.
- We study classes of decision makers, which all share some common risk preferences

Transformed wealth levels

- Consider a decision maker having initial wealth w and facing a loss X.
- Wealth after suffering the loss X :

$$w - X.$$

- Utility level after suffering the loss X

$$u\left(w - X\right).$$

  - $u\left(w - X\right)$ is a r.v.

- The expected utility is the quantity:

$$\mathbb{E}\left[u(w - X)\right].$$

Transformed wealth levels

- The expected utility hypothesis:

$$\text{Prefer loss } X \text{ over loss } Y \iff \mathbb{E}\left[u(w-X)\right] \geq \mathbb{E}\left[u(w-Y)\right],$$
$$\text{Indifferent between } X \text{ and } Y \iff \mathbb{E}\left[u(w-X)\right] = \mathbb{E}\left[u(w-Y)\right].$$

  - Relations as above hold 'provided the expectations exist'.
  - The decision maker is said to be an *expected utility maximizer*.
  - Indifferent between losses with the same distribution.
- Standardized utility functions:
  - A utility function only needs to be determined up to positive linear transformations.
    - ⋆ Exercise: prove this statement!
  - Standardize the utility function u:

$$u\left(x_0\right) = 0 \text{ and } u'\left(x_0\right) = 1, \text{for some } x_0 \in \mathbb{R}.$$

- Axiomatic framework - Von Neumann & Morgenstern (1947):
  - *Any decision maker whose behavior is in accordance with a given set of 'rational' axioms, is an expected utility maximizer.*
- The '*independence axiom*':
  - For any random losses X, Y and Z and for any Bernoulli r.v. I, independent of X, Y and Z, one has:

$$\begin{aligned} & \text{Prefer loss } X \text{ over loss } Y \\ \Rightarrow \quad & \text{Prefer loss } IX + (1-I)Z \text{ over loss } IY + (1-I)Z \end{aligned}$$

  - Example.

Definition (concave function)

A real-valued function f, defined on the interval $I \subseteq \mathbb{R}$, is **concave** on I if for any x_1, $x_2 \in I$ and any $t \in [0, 1]$,

$$f\left(tx_1 + (1-t)x_2\right) \geq tf\left(x_1\right) + (1-t)f\left(x_2\right)$$

- f is **convex** on the interval I if $(-f)$ is concave on I.
- Assume f is twice differentiable:
  - f is concave $\Leftrightarrow f''(x) \leq 0$, for all $x \in I$.
  - f is convex $\Leftrightarrow f''(x) \geq 0$, for all $x \in I$.
- f is concave $\Rightarrow f$ is continuous.

Expected utility and risk aversion

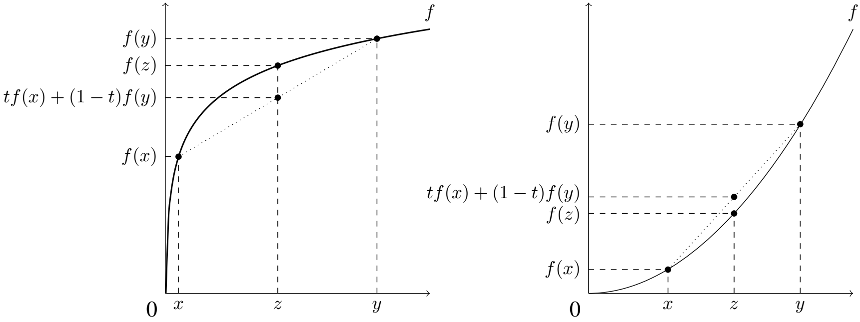


Figure: A concave (left plot) and a convex (right plot) function f where we have taken $z = tx + (1-t)y$.

Definition (Risk averse decision makers)

A decision maker is **risk averse** if his utility function u is concave on its domain.

- Risk averse decision makers have *decreasing marginal utility*.
  - Assume you gain the amount Δ.
  - Increase in utility: $u(x+\Delta) - u(x)$.
  - For risk averse decision makers, the increase in utility is a decreasing function of the wealth level x.
- Interpretation:
  - As more wealth is available, less ‘moral value’ is placed on earning an additional Euro.

Expected utility and risk aversion

- A risk is an event solely due to the whims of fate that may or may not take place
  - and that brings about some financial loss,
  - or a financial gain.
- Examples:
  - For an insurer, a risk is a potential loss (e.g. car insurance);
  - For an investor, a risk is a potential gain (e.g. investment).
- A risk always contains uncertainty:
  - The event that may or may not take place,
  - or the severity of the consequences of its occurence,
  - or the moment of its occurrence.
- Risk vs. loss:
  - 'Risk' and 'loss' are synonyms.

Theorem (Jensen's inequality (1906))

$$f \text{ is concave } \Rightarrow \mathbb{E}\left[f(Y)\right] \leq f\left(\mathbb{E}[Y]\right)$$

- Corollary: If u is a concave utility function, then

$$\mathbb{E}\left[u(w-X)\right] \leq u\left(w-\mathbb{E}[X]\right).$$

  - Exercise: prove this inequality.
- The risk averse decision maker's attitude towards risk:
  - Prefer certainty over uncertainty with the same expectation.
- The risk averse decision maker's attitude towards wealth:
  - Decreasing marginal utility.

Expected utility and risk aversion

- Definition:
  A decision maker is *risk neutral* if

$$u(x) = ax + b$$

  for given constants $a > 0$ and b.

- In this case, the expected utility hypothesis coincides with comparing expected values.

- The Arrow-Pratt measure of absolute risk aversion:

$$r(x) = \frac{-u''(x)}{u'(x)} = -\frac{d}{dx}\ln(u'(x))$$

For any risk averse decision maker, we have that $r \geq 0$.

- Risk averse individual:
  - facing a loss $X \geq 0$,
  - utility function $u(x)$,
  - initial wealth w.
- Risk averse insurer:
  - accepts X for a premium P,
  - utility function U,
  - initial wealth W.
- Under what conditions is an insurance contract feasible?
  - From the viewpoint of the individual,
  - from the viewpoint of the insurer.

- Viewpoint of the individual:
  - He is only willing to underwrite the insurance if

$$u(w-P) \geq \mathbb{E}\left[u(w-X)\right].$$

  - There exists always a premium P^M such that

$$u\left(w-P^M\right) = \mathbb{E}\left[u(w-X)\right].$$

    - ⋆ u is non-decreasing.
    - ⋆ u is concave, hence also continuous.
    - ⋆ P^M is the maximum premium the insured is willing to pay.
- From Jensen's inequality:

$$P^M \geq \mathbb{E}\left[X\right].$$

  - Exericse: prove this inequality.

- Viewpoint of the insurer:
  - He is willing to insure the risk X at a premium P if

$$U(W) \leq \mathbb{E}\left[U(W + P - X)\right].$$

  - Minimal premium P^m he requires follows from

$$U(W) = \mathbb{E}\left[U(W + P^m - X)\right].$$

  - From Jensen's inequality:

$$P^m \geq \mathbb{E}\left[X\right].$$

    - Exercise: prove this inequality.

- Condition for an insurance contract to be feasible:

$$P^m \leq P \leq P^M$$

- Definition:
  - The random vector $(X_1, X_2, \ldots, X_n)$ is said to be mutually exclusive if the following conditions hold:

$$\mathbb{P}\left[X_i \neq 0, X_j \neq 0\right] = 0, \qquad \forall i \neq j$$

- Examples of mutual exclusive couples:
  - Insurance with a franchise deductible:

$$\varphi(X) = \begin{cases} 0 \text{ if } X \leq d \\ X \text{ otherwise} \end{cases} \qquad \text{and } X - \varphi(X) = \begin{cases} X \text{ if } X \leq d \\ 0 \text{ otherwise} \end{cases}$$

  - Term insurance with doubled capital in case of accidental death.
  - Endowment insurance.

Theorem (Additivity property of mutual exclusive losses)

Consider a utility function u, satisfying $u(w) = 0$. If X and Y are mutual exclusive losses, then

$$\mathbb{E}\left[u(w - X - Y)\right] = \mathbb{E}\left[u(w - X)\right] + \mathbb{E}\left[u(w - Y)\right].$$

- A general utility function u can always be standardized such that $u(w) = 0$.
- Interpretation:
  - The utility after bearing the loss $X + Y$ is the sum of the individual expected utilities.